ADVANCED CALCULUS

WOODS

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ADVANCED CALCULUS

A COURSE ARRANGED WITH SPECIAL REFERENCE TO THE NEEDS OF STUDENTS OF APPLIED MATHEMATICS

BY

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PREFACE

The course in advanced calculus contained in this book has for many years been given by the author to students in the Massachusetts Institute of Technology. The choice of the subject matter and the arrangement of the material are the result of the experience thus gained. The students to whom the course has been given have been chiefly interested in the applications of the calculus and have felt the need of a more extensive knowledge than that gained in the elementary courses, but they have not been primarily concerned with theoretical questions. Hence there is no attempt to make this course one in analysis. However, some knowledge of theory is certainly necessary if correct use is to be made of the science; therefore the author has endeavored to introduce the students to theoretical questions and possibly to incite in some a desire for more thorough study. As an example of the method used, a proof of the existence of the definite integral in one variable has been given; for the multiple integral the proof has been omitted and simply the result stated. The student who has mastered the simpler case is in a position to read the more difficult case in easily accessible texts.

Existence proofs have also been given for the simpler cases of implicit functions and of differential equations. In these proofs the author has preferred to make the assumption that the functions involved may be expanded into Taylor series. This, of course, restricts the proof; but the somewhat immature student gets a clearer idea of the meaning of the theorems when he sees an actual series as the solution. The more abstract concept of a function may well come later. Furthermore, the student is likely to apply his results only to functions which can be expanded into series.

Because of this constant use of the power series that subject is taken up first, after certain introductory matter. Here again, following the line of simplicity, the author has not discussed series in general. The gain in concreteness for the student justifies this, but the teacher who desires to discuss series of a more general type may do so with the aid of the exercises given for the student.

The Fourier series are introduced later as tools for solving certain partial differential equations, but no attempt has been made to develop their theory.

The subjects treated in the book may be most easily seen by examining the table of contents. Experience has shown that the book may be covered in a year's course. The exercises for the student are numerous.

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ADVANCED CALCULUS

CHAPTER I

PRELIMINARY

1. Functions. *A quantity y is said to be a function of a quantity x if the value of y is determined when the value of x is given.*

Elementary examples are the familiar algebraic, trigonometric, logarithmic, and exponential functions by means of which y is explicitly given in terms of x. Such explicit formulation, however, is not necessary to the idea of a function.

For example, y may be the number of cents of postage on a letter and x the number of ounces in its weight, or y may be defined as the largest prime number which is smaller than any number x, or y may be defined as equal to 0 if x is a rational number and equal to 1 if x is an irrational number.

It should be noticed, moreover, that even when an explicit formulation in elementary functions is possible, y need not be defined by the same formula for all values of x.

For example, consider a spherical shell of inner radius a and outer radius b composed of matter of density ρ. Let x be the distance of a point from the center of the shell and y the gravitational potential due to the shell. Then y is a function of x with the following formulation:

$$\begin{aligned}
&y = 2\pi\rho(b^2 - a^2) \quad \text{when} \quad x \leqq a,\\
&y = 2\pi\rho\left(b^2 - \frac{x^2}{3}\right) - \frac{4\pi\rho}{3x}a^3 \quad \text{when} \quad a \leqq x \leqq b, \qquad (1)\\
&y = \frac{4\pi\rho}{3x}(b^3 - a^3) \quad \text{when} \quad x > b.
\end{aligned}$$

So we may at pleasure build up an arbitrary function of x. For example, let $y = f(x)$, where

$$\begin{aligned}
&f(x) = \tfrac{1}{2}x^2 \quad \text{when} \quad 0 < x < 1,\\
&f(x) = \tfrac{1}{2} \quad \text{when} \quad x = 1, \qquad (2)\\
&f(x) = \tfrac{1}{2}x + 1 \quad \text{when} \quad x > 1.
\end{aligned}$$

We shall say that values of x which lie between a and b determine an interval (a, b). The interval may or may not include the values a and b, according to the context. In general, however, the intervals (a, b) will mean the values of x defined by the statement $a \leqq x \leqq b$.

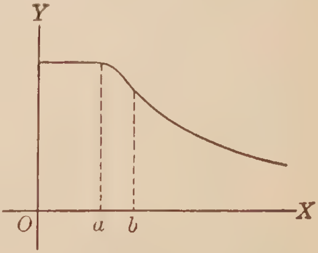

FIG. 1

The student is supposed to be familiar with the representation of a function by a graph. Such a representation is usually possible for the functions we shall handle in this book, although it is impossible for the function mentioned in the third example of this section. The interval (a, b) appears in the graph as the portion of the axis of x between $x = a$ and $x = b$, and it will be convenient to speak of a point of the interval, meaning a value of x in the interval. Then $x = a$ and $x = b$ are the end-points of the interval. As mentioned above, the interval may or may not have end-points.

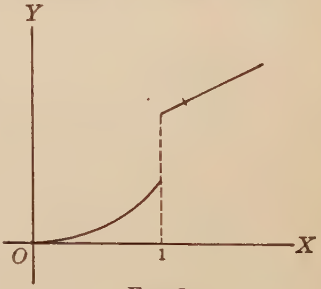

FIG. 2

The graph of the potential function in (1) is the curve of Fig. 1. The graph has no breaks and the function is continuous (§ 2), but the character of the curve and of the function is different in the three intervals considered.

The graph of the function in (2) is the curve of Fig. 2. This graph has a break at the point for which $x = 1$.

2. Continuity. A function $f(x)$ is continuous when $x = a$ for which $f(a)$ is defined if

$$\operatorname*{Lim}_{h \to 0} f[(a + h) - f(a)] = 0, \tag{1}$$

or, otherwise expressed, if

$$\operatorname*{Lim}_{h \to 0} f(a + h) = f(a), \tag{2}$$

where in either formula the limit is independent of the manner in which h approaches 0.

Since h is an increment added to a, and $f(a + h) - f(a)$ is the corresponding increment of $f(a)$, we may express this definition as follows:

A function of x is continuous for a given value of x if the increment of the function approaches zero as the increment of x approaches zero.

A more cumbersome definition, but one which brings out the full meaning of equation (1), is as follows: *$f(x)$ is continuous for $x = a$ when if ϵ is any assigned positive quantity, no matter how small, it is possible to determine another positive quantity δ so that the difference in absolute value between $f(a+h)$ and $f(a)$ shall be less than ϵ for all values of h numerically less than δ;* that is,

$$|f(a+h) - f(a)| < \epsilon \quad \text{when} \quad |h| < \delta. \tag{3}$$

Graphically, ϵ having been given, there can be found an interval $(a+h,\ a-h)$ in which $|f(x) - f(a)| < \epsilon$ at all points of the interval.

Consider the function defined by the equations

$$f(x) = \frac{10}{1 + e^{\frac{1}{x}}} \quad \text{when} \quad x \neq 0,$$
$$f(0) = 0, \tag{4}$$

the graph of which is shown in Fig. 3.

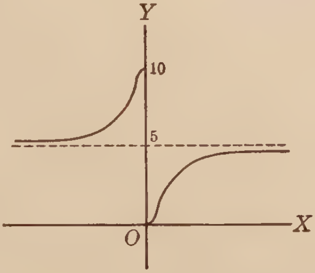

FIG. 3

Here $f(0+h) \rightarrow f(0)$ when h approaches zero through positive values, and $f(0+h) \rightarrow 10 \neq f(0)$ when h approaches zero through negative values. Hence the function is not continuous when $x = 0$. There is no interval $(-h, h)$ in which $|f(x) - f(0)| < \epsilon$. Furthermore, while the definition of $f(0)$ in (4) is arbitrary, it is not possible to define $f(0)$ so that the function is continuous.

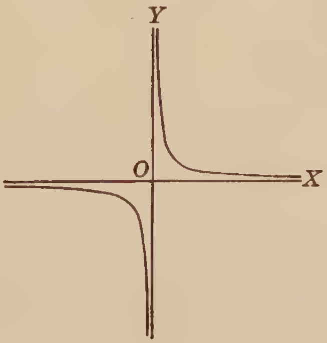

FIG. 4

It is to be noticed that $f(x)$ is not continuous for $x = a$ if $f(a)$ is "infinite." This expression means that $f(a+h)$ can be made numerically larger than any assigned positive quantity by taking h sufficiently small; or, more precisely, if M is a positive number no matter how large, then a number δ can be determined so that $|f(a+h)| > M$ for $|h| < \delta$. The definition of continuity cannot then be satisfied for $x = a$.

For example, the functions $\frac{1}{x}$ (Fig. 4) and $\frac{1}{x^2}$ (Fig. 5) are each discontinuous for $x = 0$, as is shown by the break in each of the curves representing the functions.

The following theorems are of fundamental importance in handling continuous functions:

I. If $f(x)$ is continuous at all points of an interval (a, b), it is possible to find a positive number δ such that in all *subintervals of (a, b) less than δ the absolute value of the difference between any two values of $f(x)$ is less than ϵ when ϵ is a positive quantity given in advance.*

We shall not give a formal proof. It is not difficult to see that if these theorems were not true, definition (3) for continuity must fail for at least one point of (a, b). Because of the property stated in the theorem, $f(x)$ is to be *uniformly continuous* in (a, b).

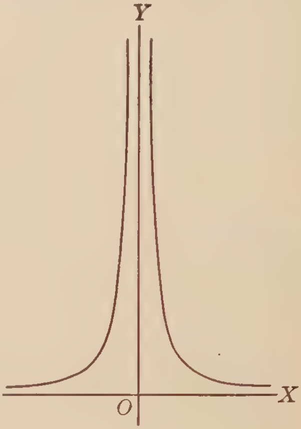

FIG. 5

II. If $f(x)$ is continuous for all values of x between a and b inclusive, if $f(a) = A$ and $f(b) = B$, and if N is any value between A and B, then $f(\xi) = N$ for at least one value of ξ between a and b.

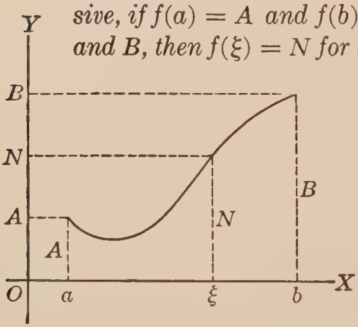

FIG. 6

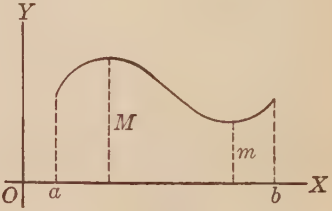

FIG. 7

III. If $f(x)$ is continuous for all values of x between a and b inclusive, then $f(x)$ has a largest value M for at least one value of x between a and b and a smallest value m for at least some other value of x between a and b.

These theorems seem to be inherent in the very nature of continuity and are graphically evident from Figs. 6, 7, and 8. As a matter of fact, however, they are not self-evident and

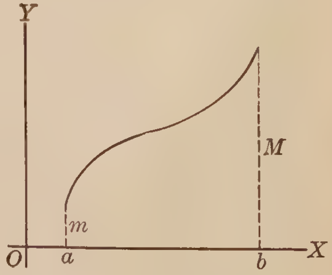

FIG. 8

are capable of rigorous proof. These proofs lie outside the range of this book and will not be given here.

The difference between the maximum and minimum values of $f(x)$ in the interval (a, b) is called the *oscillation* of $f(x)$. We can say from I,

IV. If $f(x)$ is continuous in (a, b), it is possible to find a positive number δ so that in every interval in (a, b) less than δ the oscillation of $f(x)$ is less than ϵ.

3. The derivative. A function $f(x)$ is said to have a derivative for $x = a$ if the expression

$$\frac{f(a+h)-f(a)}{h} \tag{1}$$

approaches a limit as h approaches zero in any manner whatever. This limit is called the derivative for $x = a$ and is denoted by $f'(a)$. We write

$$\operatorname*{Lim}_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a). \tag{2}$$

In order that the derivative should exist it is necessary that $f(x)$ should be continuous when $x = a$, for otherwise the fraction (1) would not approach a limit. This condition is not sufficient, as may be seen by considering the function defined by the equations

$$\begin{aligned} &f(x) = x \sin\frac{\pi}{x} \quad \text{when} \quad x \neq 0, \\ &f(0) = 0. \end{aligned} \tag{3}$$

As $x \longrightarrow 0$, $\sin\frac{\pi}{x}$ oscillates infinitely often between $+1$ and -1, but $x\sin\frac{\pi}{x} \longrightarrow 0$. Hence the function is continuous for $x = 0$. Using this function in the fraction (1) with $a = 0$, we have

$$\frac{h\sin\frac{\pi}{h} - 0}{h} = \sin\frac{\pi}{h},$$

and $\sin\frac{\pi}{h}$ does not approach a limit as $h \longrightarrow 0$. Hence the function has no derivative when $x = 0$.

In 1872 Weierstrass gave the explicit statement of a function which has for all values of x the property which $x\sin\frac{\pi}{x}$ has for $x = 0$, so that it is known now that a continuous function does not necessarily possess a derivative. Hence when a new function appears in analysis it is necessary to inquire first whether it is continuous and, secondly, whether it has a derivative. It is only functions which possess these two properties that are of interest in this book.

We have discussed the derivative of $f(x)$ for a value a of x. If $f(x)$ has a derivative at each point of an interval, there is thus defined a new function $f'(x)$ by the formula

$$f'(x) = \operatorname*{Lim}_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{4}$$

Similarly, we define $f''(x)$ as the derivative of $f'(x)$ or the second derivative of $f(x)$, $f'''(x)$ as the derivative of $f''(x)$, and so on.

It is assumed from this point that the student is familiar with the elementary process of differentiation. The proof of these elementary processes involves implicitly the proof of the continuity of the function and the existence of the derivative. The student is also assumed to be familiar with the fact that if a function is represented by a graph the derivative gives the slope of the tangent line to the graph.

The graph of the function

$$y = x \sin \frac{\pi}{x}$$

is given in Fig. 9 for positive values of x. For negative values of x the curve is reflected on the line OY. It is of course impossible to draw the curve in the close neighborhood of the point O; but it is clear that if O be joined to any other point P of the curve, the line OP oscillates through an angle of 90°. The curve therefore has no tangent line at O.

The Weierstrass function mentioned above is represented by a curve which has no breaks, but has no tangent (that is, no definite direction) at any point.

These examples illustrate the fact that a graph is at best merely a rough way to represent a function, and that conclusions drawn merely from the graph may be erroneous. The graphs are helpful in understanding or formulating a theorem, but an analytic proof is always necessary for rigor.

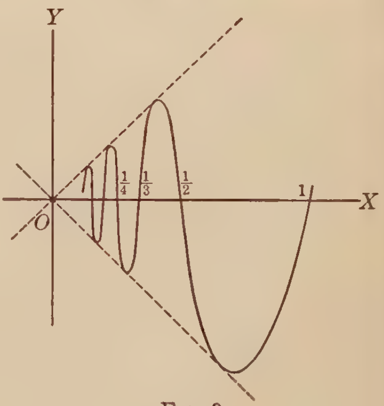

FIG. 9

4. Composite functions. Let $y = f(x)$ be a function of x and let $x = \phi(t)$. Then, by definition of a function $y = F(t)$. Let t be given an increment h and let the corresponding increment of x be k. Then

$$k = \phi(t + h) - \phi(t).$$

If $\phi(t)$ is a continuous function of t, $k \to 0$ as $h \to 0$. Now

$$F(t) = y = f(x),$$
$$F(t + h) = f(x + k),$$

since h and k are corresponding increments of t and x. Therefore

$$F(t + h) - F(t) = f(x + k) - f(x);$$

whence $$\lim_{h \to 0} [F(t + h) - F(t)] = \lim_{k \to 0} [f(x + k) - f(x)].$$

Therefore, if $f(x)$ is a continuous function,

$$\lim_{h \to 0} [F(t + h) - F(t)] = 0.$$

Hence if y is a continuous function of x and x is a continuous function of t, then y is a continuous function of t.

Let us now form the quotients

$$\frac{F(t + h) - F(t)}{h} = \frac{f(x + k) - f(x)}{h}$$
$$= \frac{f(x + k) - f(x)}{k} \cdot \frac{\phi(t + h) - \phi(t)}{h};$$

whence, by § 3, on taking the limit,

$$F'(t) = f'(x) \cdot \phi'(t). \tag{1}$$

5. Rolle's theorem. *If $f(a) = 0$, and $f(b) = 0$, then there is some value ξ between a and b for which $f'(\xi) = 0$, provided $f(x)$ is continuous in the interval $a \leqq x \leqq b$ and has a derivative for all values of x between a and b.*

By theorem III, § 2, $(f)x$ has a maximum M and a minimum m in the interval (a, b). If both M and m are zero, $f(x)$ is always zero, its derivative is zero (by (2), § 3), and the theorem is proved.

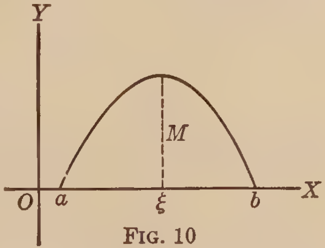

FIG. 10

Suppose that M is not zero, as in Fig. 10, and let $f(\xi) = M$. Then

$$f(\xi + h) - f(\xi)$$

is negative. Hence

$$\frac{f(\xi + h) - f(\xi)}{h}$$

is positive when h is negative, and is negative when h is positive. But, by hypothesis,

$$\frac{f(\xi+h)-f(\xi)}{h}$$

approaches a limit $f'(\xi)$, which is independent of the sign of h. Hence

$$f'(\xi)=0.$$

Again if $M=0$ but $m<0$, as in Fig. 11, the same argument applies.

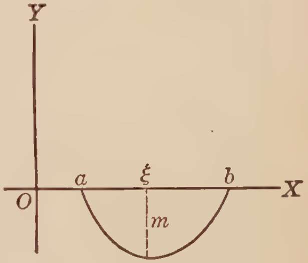

FIG. 11

The student should notice that the condition that $f(x)$ should have a derivative rules out such graphs as shown in Figs. 12 and 13, for in neither case is there a derivative in the strict sense of the definition when $x=c$. It is true that in Fig. 12 we may speak of a left-hand derivative and a right-hand derivative, but in so doing we modify the definition by first restricting h to negative values and afterwards restricting h to positive values. In Fig. 13 we may write $f'(c)=\infty$, but again the limit of $\dfrac{f(c+h)-f(c)}{h}$ does not exist in the sense of having a definite value.

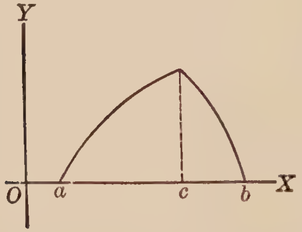

FIG. 12

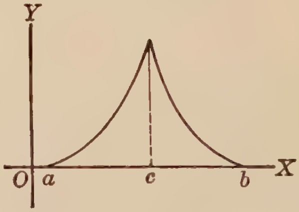

FIG. 13

6. Theorem of the mean. *I. If $f(x)$ is continuous in the interval $a \leqq x \leqq b$ and has a derivative between $x=a$ and $x=b$, then*

$$f(b)-f(a)=(b-a)f'(\xi),$$

where $a<\xi<b$.

Graphically the theorem is very obvious. In Fig. 14, $f(a)=AP$, $f(b)=BQ$, $b-a=AB$, $f(b)-f(a)=CQ$, and $\dfrac{f(b)-f(a)}{b-a}=\dfrac{CQ}{PC}=$ the slope of the chord PQ. The slope of the tangent when $x=\xi$

is $f'(\xi)$. The theorem asserts that there is a point R for which the tangent is parallel to the chord.

To prove the theorem analytically construct the auxiliary function

$$F(x) = f(x) - f(a) - (x-a)\frac{f(b)-f(a)}{b-a}.$$

Then

$$F(x+h) - F(x) = f(x+h) - f(x) - h\frac{f(b)-f(a)}{b-a}.$$

Then, if $|f(x+h) - f(x)| \to 0$ as $h \to 0$, so does $|F(x+h) - F(x)| \to 0$, and

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Now $F(a) = 0$ and $F(b) = 0$, as is seen by direct substitution. Hence, by Rolle's theorem, $F'(\xi) = 0$ for some ξ between a and b. That is,

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}. \quad (a < \xi < b)$$

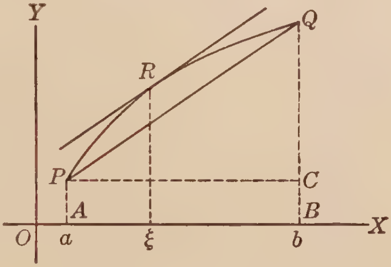

FIG. 14

From this it follows at once that

$$f(b) = f(a) + (b-a)f'(\xi), \quad (a < \xi < b) \tag{1}$$

which is the theorem to be proved.

In (1) we may write $\xi = a + \theta(b-a)$, where θ is an unknown proper fraction, and have

$$f(b) = f(a) + (b-a)f'[a + \theta(b-a)]. \quad (0 < \theta < 1) \tag{2}$$

Still another form of the same result may be obtained by placing $b = a + h$ in (2).

Then $$f(a+h) = f(a) + hf'(a + \theta h). \quad (0 < \theta < 1) \tag{3}$$

Two consequences of this theorem are as follows:

II. If the derivative of a function is zero for all values of x in an interval, the function is a constant in that interval.

In formula (1) replace b by any value of x between a and b and we have $$f(x) = f(a) + (x-a)f'(\xi). \quad (a < \xi < x)$$

But, by hypothesis, $f'(\xi) = 0$, hence

$$f(x) = f(a),$$

as was to be proved.

III. If two functions have the same derivative in an interval (a, b), *they differ by an additive constant.*

Let $f(x)$ and $\phi(x)$ be two functions such that

$$f'(x) = \phi'(x),$$

and let

$$F(x) = f(x) - \phi(x).$$

Then

$$F'(x) = f'(x) - \phi'(x) = 0;$$

whence, by II,

$$F(x) = C.$$

That is,

$$f(x) = \phi(x) + C.$$

This theorem has its important application to the process of integration, with which the student is assumed to be familiar. For let

$$\int f(x)dx$$

represent the function whose derivative is $f(x)$. Then if $F(x)$ is any one function satisfying this condition, the most general function is $F(x) + C$, and we write

$$\int f(x)dx = F(x) + C.$$

The discussion of the definite integral is postponed to Chapter VI. In the meantime we shall assume elementary knowledge when necessary.

7. Taylor's series with a remainder. The theorem of the mean is the simplest case of a more general theorem which we shall now prove. For that purpose let us write

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + R. \qquad (1)$$

This is always possible if $f(x)$ possesses the derivatives which occur in (1), since (1) itself defines R. We wish to determine the value of R as far as may be possible.

For that purpose let us define P by the relation

$$R = \frac{(b-a)^{n+1}}{(n+1)!}P, \qquad (2)$$

and write down the auxiliary function

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \cdots - \frac{(b-x)^n}{n!}f^{(n)}(x) - \frac{(b-x)^{n+1}}{(n+1)!}P, \qquad (3)$$

which is formed from (1) by changing a to x in all the terms except P.

Now $F(b) = 0$, as is at once apparent from (3), and $F(a) = 0$ by virtue of (1). Hence, by Rolle's theorem, $F'(\xi) = 0$, where ξ lies between a and b.

Differentiate (3) with respect to x. All the terms obtained cancel, except the last two, and we have

$$F'(x) = -\frac{(b-x)^n}{n!} f^{(n+1)}(x) + \frac{(b-x)^n}{n!} P. \qquad (4)$$

Substituting in (4) the value $x = \xi$, for which $F'(\xi) = 0$, we obtain

$$f^{(n+1)}(\xi) = P,$$

and therefore

$$R = \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi). \qquad (5)$$

This is the value of R in (1). It measures the difference between the value of $f(b)$ and the sum of the first $n+1$ terms in the right-hand member of (1). It may therefore be called the *remainder* after $n+1$ terms.

The formula has great theoretical value. In addition it may be used in calculation as follows:

If we know the values of $f(x)$ and its derivatives for $x = a$, we may compute the value of $f(x)$ for $x = b$ by use of the first $n+1$ terms of (1). The quantity R as given in (5) will then measure the error made in taking the result of this calculation for the value of $f(b)$. It is true that the value of R will not be exactly known, since ξ is unknown, but it may frequently be possible to determine a numerical quantity which R cannot exceed in absolute value.

For example, consider $\sin x$. The values of $\sin x$ and $\cos x$ are assumed to be known for $x = a = \frac{\pi}{3}$. We wish to find the value of $\sin x$ for $x = b = \frac{\pi}{3} + \frac{\pi}{180} = \frac{61}{180}\pi$. Expressed in degrees we need to find sin 61° knowing the sine and cosine of 60°.

From (1) we have

$$\sin\frac{61\pi}{180} = \sin\frac{\pi}{3} + \frac{\pi}{180}\cos\frac{\pi}{3} - \frac{1}{2!}\left(\frac{\pi}{180}\right)^2 \sin\frac{\pi}{3} - \frac{1}{3!}\left(\frac{\pi}{180}\right)^3 \cos\frac{\pi}{3} + R,$$

where

$$R = \frac{1}{4!}\left(\frac{\pi}{180}\right)^4 \sin\xi. \quad \left(\frac{\pi}{3} < \xi < \frac{61\pi}{180}\right)$$

Since we know that the sine of any angle is less than unity, we have

$$R < \frac{1}{4!}\left(\frac{\pi}{180}\right)^4 < .000000004.$$

As another example let us attempt to find how many terms of (1) must be taken to compute sin 12° to the nearest sixth decimal place. In (1) we place $a = 0$, $b = \frac{12\,\pi}{180} = \frac{\pi}{15}\cdot$ Then

$$\sin\frac{\pi}{15} = \frac{\pi}{15} - \frac{1}{3!}\left(\frac{\pi}{15}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{15}\right)^5 - \cdots \pm \frac{1}{n!}\left(\frac{\pi}{15}\right)^n + R. \qquad (6)$$

Here R is $\pm\frac{1}{(n+1)!}\left(\frac{\pi}{15}\right)^{n+1}\sin\xi$, with $0 < \xi < \frac{\pi}{15}$, and we are sure that

$$|\,R\,| < \frac{1}{(n+1)!}\left(\frac{\pi}{15}\right)^{n+1},$$

and we wish to determine n so that

$$\frac{1}{(n+1)!}\left(\frac{\pi}{15}\right)^{n+1} < .0000005.$$

By trial we find that $n+1 = 7$, $n = 6$, and hence the first three terms of (6) are to be taken. This value of n is sufficient for the purpose, but in this case it is not necessary. From the manner in which n was obtained it is clear that a smaller value of n may do.

Since we have used Rolle's theorem in deriving (1), the hypotheses underlying that theorem must be met; that is, $F(x)$ in (2) should be continuous and possess a derivative in the interval (a, b). This means that $f(x)$ and its first $n+1$ derivatives should each exist and be continuous in the same interval. Then if x is any value in that interval, the same conditions exist in the interval (a, x), and we may write in place of (1)

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$
$$+ \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi). \quad (a < \xi < x) \qquad (7)$$

Equations (1) and (7) are two forms of *Taylor's series with the remainder.*

In the particular case in which $a = 0$, we have *Maclaurin's series with the remainder*; namely,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi). \qquad (0 < \xi < x) \tag{8}$$

The student has probably met in his elementary course the infinite series known as Taylor's and Maclaurin's series. He should therefore note carefully that we have to do here not with infinite series, but with finite polynomials, although the last term is not definitely known. The infinite series arise from (7) or (8) if n can be taken indefinitely great and if the value of R approaches zero as n increases without limit. The discussion of this case, however, leads to questions of convergence and the like, and will be postponed. As a matter of fact, the finite series (7) and (8) are sufficient for most practical purposes.

For example, we find from (8)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}f^{(7)}(\xi)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} + Px^7,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}f^{(6)}(\xi)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + Qx^6.$$

Therefore $$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} + Px^7}{1 - \frac{x^2}{2} + \frac{x^4}{24} + Qx^6}$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + Sx^7,$$

where S is determined in terms of P and Q by the usual process of division. This is a much simpler way to expand $\tan x$ than by direct use of (8).

Again, consider $\displaystyle\int_0^x \frac{dx}{1+x^2} = \tan^{-1} x.$

We may write

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \pm x^{2k} \mp x^{2k+2}P.$$

Therefore

$$\int_0^x \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \pm \frac{x^{2k+1}}{2k+1} \mp \int_0^x x^{2k+2}P\,dx.$$

Now $P = \dfrac{1}{1+x^2}$ by direct division. Therefore

$$P < 1 \quad \text{and} \quad \int_0^x x^{2k+2}P\,dx < \int_0^x x^{2k+2}dx < \frac{x^{2k+3}}{2k+3}.$$

We have, therefore,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \pm \frac{x^{2k+1}}{2k+1} + R,$$

where

$$|R| < \frac{x^{2k+3}}{2k+3}.$$

Other forms of the remainder besides that given in (5) are also useful. If we apply Rolle's theorem to the function

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \cdots - \frac{(b-x)^n}{n!}f^{(n)}(x) - \frac{b-x}{b-a}R,$$

which vanishes when $x = a$ and when $x = b$, we find that

$$R = \frac{(b-a)(b-\xi)^n}{n!}f^{(n+1)}(\xi), \tag{9}$$

or, writing $\xi = a + \theta(b-a)$, where $0 < \theta < 1$,

$$R = \frac{(b-a)^{n+1}(1-\theta)^n}{n!}f^{(n+1)}(\xi). \tag{10}$$

Again, if we start with the identity

$$f(x) - f(a) = \int_0^{x-a} f'(x-t)dt$$

and integrate by parts, we have

$$f(x) - f(a) = (x-a)f'(a) + \int_0^{x-a} tf''(x-t)dt.$$

Again integrating by parts, we have

$$f(x) - f(a) = (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \int_0^{x-a}\frac{t^2}{2!}f'''(x-t)dt.$$

Proceeding in this way, we have, finally

$$R = \int_0^{x-a} \frac{t^n}{n!} f^{(n+1)}(x-t)dt. \tag{11}$$

In particular for Maclaurin's series, where $a = 0$,

$$R = \int_0^{x} \frac{t^n}{n!} f^{(n+1)}(x-t)dt. \tag{12}$$

8. The form $\frac{0}{0}$. Consider the fraction

$$\frac{f(x)}{\phi(x)},$$

and let there be a number a for which $f(a) = 0$ and $\phi(a) = 0$. The substitution of $x = a$ in the fraction produces the meaningless symbol $\frac{0}{0}$, so that the value of the fraction is not defined for $x = a$. It is customary, however, to extend the definition of the fraction by defining its value for $x = a$ as the limit approached by its value as x approaches a. For example, consider

$$\frac{x^2 - a^2}{x - a}.$$

For all values of x except $x = a$ the value of this fraction is $x + a$. As $x \to a$, $x + a \to 2a$; therefore we say, *by definition*, that the value of $\frac{x^2 - a^2}{x - a}$ when $x = a$ is $2a$.

To obtain a general method for finding this limit we begin by applying Rolle's theorem to the function

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)}[\phi(x) - \phi(a)] - [f(x) - f(a)],$$

which obviously vanishes when $x = a$ and when $x = b$.

Hence
$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(\xi)}{\phi'(\xi)}. \quad (a < \xi < b) \tag{1}$$

In (1) let $f(a) = 0$, $\phi(a) = 0$, and $b = x$. We have

$$\frac{f(x)}{\phi(x)} = \frac{f'(\xi)}{\phi'(\xi)}. \quad (a < \xi < x) \tag{2}$$

Now as $x \to a$, $\xi \to a$, and therefore

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{\xi \to a} \frac{f'(\xi)}{\phi'(\xi)}. \tag{3}$$

Now unless $f'(a) = 0$ and $\phi'(a) = 0$ we have the result

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}. \tag{4}$$

If, however, $f'(a) = 0$ and $\phi'(a) = 0$, we must apply (3) again with the result

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{\xi \to a} \frac{f'(\xi)}{\phi'(\xi)} = \frac{f''(a)}{\phi''(a)}$$

unless $f''(a) = 0$ and $\phi''(a) = 0$. In the latter case formula (3) must be applied again.

We may sum up in a rule known as *L'Hospital's rule.*

To find the value of a fraction which takes the form $\frac{0}{0}$ *when* $x = a$, *replace the numerator and the denominator each by its derivative and substitute* $x = a$. *If the new fraction is also* $\frac{0}{0}$, *repeat the process.*

For example,

$$\lim_{x \to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x} = \lim_{x \to 0} \frac{e^x + 2\sin x - e^{-x}}{\sin x + x\cos x} = \left[\frac{e^x + 2\cos x + e^{-x}}{2\cos x - x\sin x}\right]_{x=0} = 2.$$

9. The form $\frac{\infty}{\infty}$. Consider the fraction

$$\frac{f(x)}{\phi(x)},$$

and let $f(a) = \infty$ and $\phi(a) = \infty$ by hypothesis. The value of the fraction for $x = a$ is then defined as the limit approached by the value of the fraction as x increases without limit.

We shall prove that *L'Hospital's rule holds also for a fraction which takes the form* $\frac{\infty}{\infty}$.

We shall first take the case in which $a = \infty$.

From (1), § 8, we may write

$$\frac{f(x) - f(c)}{\phi(x) - \phi(c)} = \frac{f'(\xi)}{\phi'(\xi)}, \quad (c < \xi < x) \tag{1}$$

where c is a large but finite value of x. From (1) we derive, by simple algebra,

$$\frac{f(x)}{\phi(x)} = \frac{f'(\xi)}{\phi'(\xi)} \cdot \frac{1 - \dfrac{\phi(c)}{\phi(x)}}{1 - \dfrac{f(c)}{f(x)}}. \tag{2}$$

We shall now assume that $\frac{f'(\xi)}{\phi'(\xi)}$ has a limit A as $\xi \to \infty$. We may consequently take c so large that $\frac{f'(c)}{\phi'(c)}$ and therefore $\frac{f'(\xi)}{\phi'(\xi)}$ differs from A by less than any assigned positive quantity ϵ_1. This fixes c. Then $f(c)$ and $\phi(c)$ are finite, and x may be taken so large that

$$\frac{1-\dfrac{\phi(c)}{\phi(x)}}{1-\dfrac{f(c)}{f(x)}}$$

differs from unity by less than any assigned positive quantity ϵ_2. We then have, from (2),

$$\frac{f(x)}{\phi(x)} = (A+\eta_1)(1+\eta_2),$$

where $$|\eta_1| < \epsilon_1, \qquad |\eta_2| < \epsilon_2.$$

From this it is apparent that

$$\operatorname*{Lim}_{x\to\infty} \frac{f(x)}{\phi(x)} = A = \operatorname*{Lim}_{x\to\infty} \frac{f'(\xi)}{\phi'(\xi)} = \operatorname*{Lim}_{x\to\infty} \frac{f'(x)}{\phi'(x)}. \tag{3}$$

This justifies L'Hospital's rule for the case $a = \infty$. We need now to extend this result to the case in which $\frac{f(x)}{\phi(x)}$ becomes $\frac{\infty}{\infty}$ for $x = a$ when a is not infinite. For that purpose place $x = a + \frac{1}{y}$ so that when $x = a$, $y = \infty$. Then

$$\frac{f(x)}{\phi(x)} = \frac{f\left(a+\dfrac{1}{y}\right)}{\phi\left(a+\dfrac{1}{y}\right)} = \frac{F(y)}{\Phi(y)}, \tag{4}$$

and $$\operatorname*{Lim}_{x\to a} \frac{f(x)}{\phi(x)} = \operatorname*{Lim}_{y\to\infty} \frac{F(y)}{\Phi(y)} = \operatorname*{Lim}_{y\to\infty} \frac{F'(y)}{\Phi'(y)}. \tag{5}$$

But $$F'(y) = f'(x)\frac{dx}{dy} = \frac{-1}{y^2} f'(x),$$

and $$\Phi'(y) = \phi'(x)\frac{dx}{dy} = \frac{-1}{y^2}\phi'(x).$$

Therefore (5) gives $$\operatorname*{Lim}_{x\to a} \frac{f(x)}{\phi(x)} = \operatorname*{Lim}_{x\to a} \frac{f'(x)}{\phi'(x)}. \tag{6}$$

From results (3) and (6) L'Hospital's rule follows.

10. Other indeterminate forms. A product

$$f(x) \cdot \phi(x) \tag{1}$$

may give rise for a value of $x = a$ to a form

$$0 \cdot \infty,$$

and a difference

$$f(x) - \phi(x) \tag{2}$$

may give rise when $x = a$ to a form

$$\infty - \infty.$$

In such cases it is usually possible by an elementary operation to transform the product (1) or the difference (2) to a fraction which takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when $x = a$, and apply L'Hospital's rule.

For example, the product $x^n e^{-x^2}$

when n is positive becomes $\infty \cdot 0$ for $x = \infty$.

We have, however,

$$x^n e^{-x^2} = \frac{x^n}{e^{x^2}}.$$

Then, by L'Hospital's rule,

$$\lim_{x \to \infty} \frac{x^n}{e^{x^2}} = \lim_{x \to \infty} \frac{nx^{n-2}}{2\,e^{x^2}} = \lim_{x \to \infty} \frac{n(n-2)x^{n-3}}{4\,e^{x^2}}.$$

Proceeding in this way it appears that eventually x will disappear from the numerator of the fraction, no matter what n is, and therefore

$$\lim_{x \to \infty} x^n e^{-x^2} = 0.$$

Again, the difference $\sec x - \tan x$ becomes $\infty - \infty$ when $x = \frac{\pi}{2}$.

But

$$\sec x - \tan x = \frac{1 - \sin x}{\cos x},$$

which becomes $\frac{0}{0}$ when $x = \frac{\pi}{2}$. Therefore

$$\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x) = \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} = 0.$$

An expression of the type

$$[f(x)]^{\phi(x)}$$

may, when $x = a$, give rise to forms

$$0^0, \quad \infty^0, \quad 0^\infty, \quad 1^\infty,$$

and the like. Such an expression may be reduced to a type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by the use of logarithms. Thus, we write

$$u = [f(x)]^{\phi(x)}.$$

Then $$\log u = \phi(x) \cdot \log f(x).$$

If $\lim\limits_{x \to a} \phi(x) \cdot \log f(x)$ can be found by the previous methods, the limit approached by u can be found.

Consider as an example $\quad (1-x)^{\frac{1}{x}}$

when $x = 0$. A plausible procedure would be to place $x = 0$ and, obtaining 1^{∞}, to say that this is 1 since any power of 1 is 1. But this would ignore the fact that we are interested in the *limit* of $(1-x)^{\frac{1}{x}}$ as x approaches 0, which we have defined as the value of the fraction when $x = 0$. We therefore write

$$u = (1-x)^{\frac{1}{x}},$$

$$\log u = \frac{1}{x} \log (1-x) = \frac{\log (1-x)}{x}.$$

By L'Hospital's rule,

$$\lim_{x \to 0} \frac{\log (1-x)}{x} = -1.$$

Therefore $$\lim_{x \to 0} \log u = -1,$$

and $$\lim_{x \to 0} u = e^{-1}.$$

11. Infinitesimals. *An infinitesimal is defined as a variable which approaches zero as a limit.* When two or more infinitesimals approach zero at the same time, they may be compared by considering their ratios.

We say an infinitesimal β is of the *same order* as an infinitesimal α if

$$\lim \frac{\beta}{\alpha} = k, \tag{1}$$

where k is a finite quantity different from zero.

An infinitesimal β is of *higher order* than an infinitesimal α if

$$\lim \frac{\beta}{\alpha} = 0. \tag{2}$$

As an example, consider an infinitesimal angle α (represented in Fig. 15) and describe an arc of a circle PQ of radius unity. Let

$$\beta = PN = \sin \alpha,$$
$$\gamma = NQ = 1 - \cos \alpha.$$

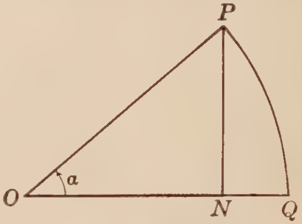

FIG. 15

By a well-known theorem which is used in deriving the formula for the derivative of $\sin x$,

$$\operatorname{Lim} \frac{\beta}{\alpha} = 1.$$

Therefore PN is of the same order as α.

Also $$\operatorname{Lim} \frac{\gamma}{\alpha} = \operatorname{Lim} \frac{1 - \cos \alpha}{\alpha} = 0.$$

Hence γ is of higher order than α.

A measure of an order of an infinitesimal may be given as follows. If α is taken as an infinitesimal of the first order, then α^2, α^3, α^4, $\cdots$ are called infinitesimals of the second, third, and fourth orders, respectively, and β is an infinitesimal of nth order with respect to α, where n is positive, if

$$\operatorname*{Lim}_{\alpha \to 0} \frac{\beta}{\alpha^n} = k, \tag{3}$$

where k is a finite quantity not zero.

For example, consider $\gamma = NQ$ of Fig. 15. We have

$$\operatorname*{Lim}_{\alpha \to 0} \frac{1 - \cos \alpha}{\alpha^2} = \frac{1}{2}.$$

Therefore NQ is of the second order with respect to α.

Equations (1) and (3) lead to the forms

$$\beta = k\alpha + \alpha\epsilon, \tag{4}$$
$$\beta = k\alpha^n + \alpha^n\epsilon, \tag{5}$$

respectively, where ϵ is another infinitesimal. In each case the first term on the right of the equation is called the *principal* part of the infinitesimal. An infinitesimal and its principal part are obviously of the same order and differ by an infinitesimal of higher order than either of them.

If we denote by β_1 the principal part of β, formulas (4) and (5) become

$$\beta = \beta_1 + \beta_1 \frac{\epsilon}{k} = \beta_1 + \beta_1 \epsilon_1. \tag{6}$$

There are two convenient ways to determine the order of an infinitesimal β with respect to α: One is to evaluate

$$\operatorname*{Lim}_{\alpha \to 0} \frac{\beta}{\alpha^n}$$

by L'Hospital's rule, so choosing n that the limit is finite and not zero. A more expeditious way, when β can be placed equal to a function of α, is to expand $\beta = f(\alpha)$ by Maclaurin's series; then, if

$$\beta = k\alpha^n + k_1\alpha^{n+1} + R = k\alpha^n + k_1\alpha^{n+1} + \alpha^{n+2}P,$$

we have
$$\frac{\beta}{\alpha^n} = k + k_1\alpha + \alpha^2 P$$

and
$$\operatorname*{Lim}_{\alpha \to 0} \frac{\beta}{\alpha^n} = k,$$

which shows that the degree of the first term in the expansion of β determines the order of β.

We shall illustrate these methods by inquiring by what order of infinitesimals an infinitesimal arc of a circle exceeds its chord. In a circle of radius a and center O (Fig. 16) let $\widehat{AB}$ be an infinitesimal arc and $\overline{AB}$ its infinitesimal chord. Draw the radii OA and OB, and draw ODC perpendicular to $\overline{AB}$. Let the angle $BOC = \theta$.

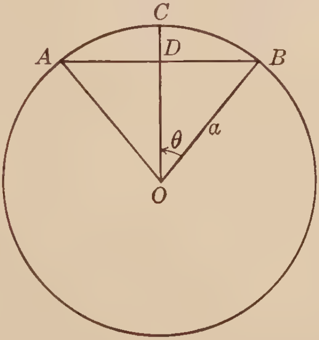

FIG. 16

Then
$$\widehat{AB} = 2\,a\theta,$$
$$\overline{AB} = 2\,a \sin\theta,$$

and
$$\widehat{AB} - \overline{AB} = 2\,a\theta - 2\,a\sin\theta.$$

Take $\widehat{AB}$ as the α of our general discussion, and let

$$\beta = \widehat{AB} - \overline{AB}.$$

We have
$$\operatorname*{Lim}_{\alpha \to 0} \frac{\beta}{\alpha} = \operatorname*{Lim}_{\theta \to 0} \frac{2\,a\theta - 2\,a\sin\theta}{2\,a\theta} = 0.$$

Hence β is of higher order than α.

Similarly,
$$\operatorname*{Lim}_{\alpha \to 0} \frac{\beta}{\alpha^2} = 0, \quad \operatorname*{Lim}_{\alpha \to 0} \frac{\beta}{\alpha^3} = \frac{1}{24\,a^2},$$

and therefore β is of the third order with respect to α.

The second method is to place

$$\begin{aligned}\beta &= 2\,a\theta - 2\,a\sin\theta\\ &= 2\,a\theta - 2\,a\left(\theta - \frac{\theta^3}{3!} + R\right)\\ &= \frac{2\,a\theta^3}{3!} + R'\\ &= \frac{\alpha^3}{24\,a^2} + R'.\end{aligned}$$

We have accordingly shown in two ways that the difference between an infinitesimal arc of a circle and its chord is an infinitesimal of the third order.

12. Fundamental theorems on infinitesimals. There are two important theorems involving infinitesimals; namely,

I. If the quotient of two infinitesimals has a limit, that limit is unaltered by replacing either infinitesimal by its principal part.

To prove this let us place

$$\beta = \beta_1 + \beta_1\epsilon_1, \quad \alpha = \alpha_1 + \alpha_1\epsilon_2,$$

in accordance with (6), § 11. Then

$$\frac{\beta}{\alpha} = \frac{\beta_1 + \beta_1\epsilon_1}{\alpha_1 + \alpha_1\epsilon_2} = \frac{\beta_1}{\alpha_1}\,\frac{1+\epsilon_1}{1+\epsilon_2};$$

whence

$$\text{Lim}\,\frac{\beta}{\alpha} = \text{Lim}\,\frac{\beta_1}{\alpha_1}. \tag{1}$$

II. If the sum of n positive infinitesimals has a limit as n increases indefinitely and each infinitesimal approaches zero, that limit is unaltered by replacing each infinitesimal by its principal part.

Let $\beta_1, \beta_2, \cdots, \beta_n$ be a set of n positive infinitesimals, and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be their principal parts, also positive. Then

$$\beta_i = \alpha_i + \alpha_i\epsilon_i,$$

and

$$\sum\beta_i = \sum\alpha_i + \sum\alpha_i\epsilon_i. \tag{2}$$

Let γ be a positive quantity which is equal to the largest absolute value of ϵ_i. Then, for any i,

$$-\gamma \leqq \epsilon_i \leqq \gamma,$$

and

$$-\gamma\alpha_i \leqq \epsilon_i\alpha_i \leqq \gamma\alpha_i,$$

the multiplication being allowable since α_i is positive.

Then

$$-\gamma\sum\alpha_i \leqq \sum\epsilon_i\alpha_i \leqq \gamma\sum\alpha_i. \tag{3}$$

By hypothesis, as n increases indefinitely, $\Sigma\alpha_i$ approaches a finite limit and γ approaches zero. Therefore, from (3),

$$\operatorname{Lim} \sum \alpha_i \epsilon_i = 0,$$

and therefore, from (2), $\operatorname{Lim} \sum \beta_i = \operatorname{Lim} \sum \alpha_i.$

The proof uses the assumption that all the infinitesimals are positive. The theorem is also true if all the infinitesimals are negative, since the sum of such infinitesimals is minus the sum of a set of positive infinitesimals. The theorem is not necessarily true if the infinitesimals are not all of the same sign. A single example will show this.

Let $\alpha_1 = \frac{1}{\sqrt{n}}, \quad \alpha_2 = -\frac{1}{\sqrt{n}}, \quad \alpha_3 = \frac{1}{\sqrt{n}}, \quad \alpha_4 = -\frac{1}{\sqrt{n}}, \quad \cdots,$
and $\beta_1 = \alpha_1 + \frac{1}{\sqrt{n}}\alpha_1, \quad \beta_2 = \alpha_2 - \frac{1}{\sqrt{n}}\alpha_2, \quad \beta_3 = \alpha_3 + \frac{1}{\sqrt{n}}\alpha_3,$
$\beta_4 = \alpha_4 - \frac{1}{\sqrt{n}}\alpha_4, \quad \cdots;$ whence $\beta_1 = \alpha_1 + \frac{1}{n}, \quad \beta_2 = \alpha_2 + \frac{1}{n},$
$\beta_3 = \alpha_3 + \frac{1}{n}, \ \beta_4 = \alpha_4 + \frac{1}{n},$ and so on.

Now, by direct summation,

$$\sum \alpha_i = 0 \text{ or } \frac{1}{\sqrt{n}}$$

according as n is even or odd, and in each case

$$\operatorname*{Lim}_{n \to \infty} \sum \alpha_i = 0.$$

Also $$\sum \beta_i = \sum \alpha_i + 1,$$

and therefore $$\operatorname{Lim} \sum \beta_i = 1.$$

13. Some geometric theorems involving infinitesimals. We shall give in this section certain geometric theorems which are of some importance in our subsequent work.

I. Under certain general hypotheses the length of an infinitesimal arc differs from that of its chord by an infinitesimal of higher order than either.

Consider an arc of a curve $\widehat{AB}$ (Fig. 17) and its chord $\overline{AB}$. We shall assume that the arc is a continuous curve and has a continuously changing direction.

This amounts to saying that if axes of x and y are chosen by which the equation of the curve becomes $y = f(x)$, $f(x)$ is a continuous function of x with a continuous derivative.

We shall also assume that a perpendicular from any point P of the arc $\overset{\frown}{AB}$ meets the chord $\overline{AB}$ in one and only one point Q, and that as P moves continuously from A to B along the arc, Q moves continuously and always in one direction along the chord.

The length of the arc $\overset{\frown}{AB}$ is defined as follows:

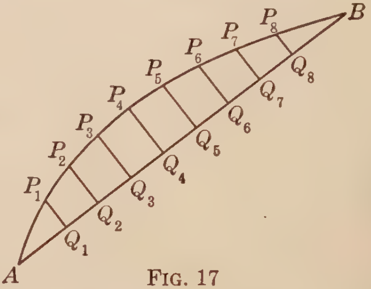

FIG. 17

The arc is divided in any manner into n parts, and chords are drawn connecting the points of division. If the sum of the lengths of these chords approaches a limit, independently of the manner of division, as n is indefinitely increased while the length of each chord approaches zero, that limit is by definition the length of $\overset{\frown}{AB}$. We shall assume that the arc $\overset{\frown}{AB}$ has a length.

Divide the arc $\overset{\frown}{AB}$ into n parts by the points $P_1, P_2, \cdots, P_{n-1}$, draw the chords $P_iP_{i+1} (P_0 = A,\ P_n = B)$, and drop perpendiculars from the points P_i to the chord $\overline{AB}$, thus determining the points $Q_1, Q_2, \cdots, Q_{n-1}$. Let the lengths of the chords $AP_1, P_1P_2, \cdots, P_{n-1}B$ be $\alpha_1, \alpha_2, \cdots, \alpha_n$ and the lengths of the segments $AQ_1, Q_1Q_2, \cdots, Q_{n-1}B$ be $\beta_1, \beta_2, \cdots, \beta_n$.

Then, if l is the length of the chord $\overline{AB}$ and s the length of the arc $\overset{\frown}{AB}$, which by hypothesis exists,

$$\beta_1 + \beta_2 + \cdots + \beta_n = l, \tag{1}$$

$$\lim_{n \to \infty} (\alpha_1 + \alpha_2 + \cdots + \alpha_n) = s. \tag{2}$$

Now β_i is the projection of α_i on $\overline{AB}$. Hence, by the law of projections,

$$\beta_i = \alpha_i \cos \theta_i,$$

where θ_i is the angle between the chord of length α_i and the chord $\overline{AB}$. Hence (2) is

$$\lim_{n \to \infty} (\beta_1 \sec \theta_1 + \beta_2 \sec \theta_2 + \cdots + \beta_n \sec \theta_n) = s. \tag{3}$$

Under our hypotheses $\sec \theta_i$ is always positive, although θ_i may be negative. Our hypotheses allow us to apply the theorem of the mean to any portion of the curve between A and B. Therefore there is some tangent which makes an angle θ_i with $\overline{AB}$. Hence, if ϕ is the largest angle in absolute value which any tangent makes with $\overline{AB}$, we have

$$1 \leqq \sec \theta_i \leqq \sec \phi.$$

Therefore

$$\beta_1 + \beta_2 + \cdots + \beta_n \leqq \beta_1 \sec \theta_1 + \beta_2 \sec \theta_2 + \cdots + \beta_n \sec \theta_n$$
$$\leqq \sec \phi (\beta_1 + \beta_2 + \cdots + \beta_n),$$

or, from (1) and from (3),

$$l \leqq s \leqq l \sec \phi, \tag{4}$$

or

$$1 \leqq \frac{s}{l} \leqq \sec \phi, \tag{5}$$

where the equality signs could hold only if $\widehat{AB}$ coincided with $\overline{AB}$.

This result is true for any finite arc for which the hypotheses that have been made hold. It remains true as B approaches A. But then $\sec \phi$ approaches unity. Hence we have

$$\lim_{l \to 0} \frac{s}{l} = 1, \tag{6}$$

or

$$s = l + l\epsilon,$$

which was to be proved.

II. Under the hypothesis made in I the perpendicular distance from one end of an infinitesimal arc to the tangent at the other end is an infinitesimal of higher order than the arc, and the length of the tangent from the foot of this perpendicular to the point of tangency is an infinitesimal of the same order.

Consider again the arc $\widehat{AB}$ (Fig. 18) with the properties as before. Draw the tangent AT at A, and drop the perpendicular BT. Let $AT = t$ and $BT = h$ and angle $BAT = \alpha$.

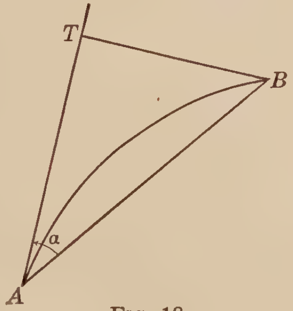

FIG. 18

Then $t = l \cos \alpha, \qquad h = l \sin \alpha,$

and $\dfrac{t}{s} = \dfrac{l}{s} \cos \alpha, \qquad \dfrac{h}{s} = \dfrac{l}{s} \sin \alpha.$

Let $s \to 0$ and apply (6). We have

$$\lim_{s \to 0} \frac{t}{s} = 1, \qquad \lim_{s \to 0} \frac{h}{s} = 0. \tag{7}$$

The relation between t and h may also be found by means of coördinate axes and Maclaurin's series.

Take as origin a point on the curve (Fig. 19) and as OX the tangent at O. Let

$$y = f(x)$$

be the equation of the curve. Then $f(0) = 0$ and $f'(0) = 0$, since the curve passes through O and the slope of the tangent at O is zero. Then, by Maclaurin's series,

$$y = f(x) = f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + R.$$

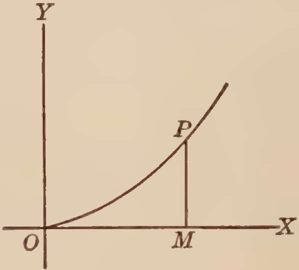

FIG. 19

But $OM = x$ and $MP = y$. Hence it appears that MP is of the second order with respect to OM unless $f''(0) = 0$.

III. Except for infinitesimals of higher order than the lengths of the arcs, an infinitesimal right-angled curvilinear triangle obeys the same trigonometric laws as a straight-lined right-angled triangle when the hypotheses of I are satisfied.

Consider a triangle ABC (Fig. 20) whose sides are arcs of curves which satisfy the hypotheses of I and which may be made to approach zero together. Let the arcs intersect at a right angle at C and let the angle at A be ϕ. Draw the chords $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. The angle between the chords $\overline{AB}$ and $\overline{AC}$ is $\phi + \epsilon_1$ and that between $\overline{BC}$ and $\overline{CA}$ is $90° + \epsilon_2$, where ϵ_1 and ϵ_2 are infinitesimals approaching zero as the sides of the curvilinear triangle approach zero. Then

$$\frac{\overline{BC}}{\overline{AB}} = \frac{\sin(\phi + \epsilon_1)}{\sin(90° + \epsilon_2)},$$

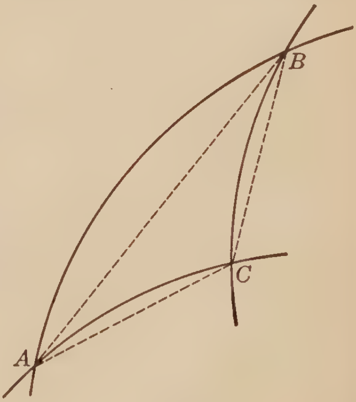

FIG. 20

which we may write as

$$\frac{\widehat{BC}}{\widehat{AB}} \cdot \frac{\widehat{AB}}{\overline{AB}} \cdot \frac{\overline{BC}}{\widehat{BC}} = \frac{\sin(\phi + \epsilon_1)}{\sin(90° + \epsilon_2)}$$

where $\widehat{BC}$ means the arc BC, and so on.

Taking the limit as A, B, and C approach coincidence, and using theorem I, we have

$$\operatorname{Lim} \frac{\overset{\frown}{BC}}{\overset{\frown}{AB}} = \sin \phi,$$

or

$$\overset{\frown}{BC} = \overset{\frown}{AB} \sin \phi + \epsilon_3 \overset{\frown}{AB}.$$

In a similar manner,

$$\overset{\frown}{AC} = \overset{\frown}{AB} \cos \phi + \epsilon_4 \overset{\frown}{AB},$$

$$\overset{\frown}{BC} = \overset{\frown}{AB} \tan \phi + \epsilon_5 \overset{\frown}{AB},$$

$$\overset{\frown}{AB}^2 = \overset{\frown}{AC}^2 + \overset{\frown}{BC}^2 + \epsilon_6 \overset{\frown}{AB}.$$

As an example of the use of the foregoing theorem consider an ellipse with foci F and F' (Fig. 21).

Let P and Q be two points infinitesimally near on the ellipse, and draw PF, PF', QF, and QF'. By the definition of the ellipse,

$$PF + PF' = QF + QF'.$$

With F as a center and a radius FQ construct an arc of a circle cutting FP in S. With F' as a center and a radius $F'Q$ construct an arc of a circle cutting $F'P$ in R.

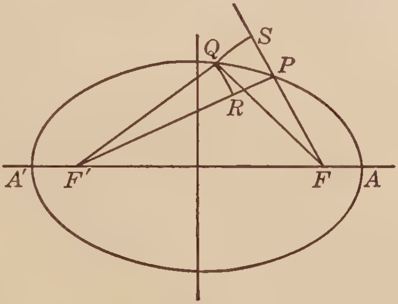

FIG. 21

Then $\quad SP - RP = QF - PF - (PF' - QF') = 0.$

Then in the infinitesimal triangles SQP and RQP

$$SP = QP \cos SPQ,$$

$$RP = QP \cos RPQ,$$

except for infinitesimals of higher order. Therefore

$$\cos SPQ = \cos RPQ$$

except possibly for infinitesimals of higher order.

But the angles SPQ and RPQ are independent of the position of Q.

Hence $\qquad SPQ = RPQ;$

and, since $\qquad SPQ = FPA,$

we have the result that in an ellipse the tangent at any point makes equal angles with the focal radii to that point.

14. The first differential. Consider now a function $f(x)$ which has a derivative $f'(x)$. If Δx is an infinitesimal increment of x, then the increment $\Delta y = f(x + \Delta x) - f(x)$ is an infinitesimal, since $f(x)$ is continuous. Now

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x);$$

whence

$$\Delta y = f'(x)\Delta x + \epsilon\,\Delta x. \qquad (1)$$

Aside from the values of x for which $f'(x)$ is zero, or infinite, this is of the form (4), § 11, and $f'(x)\Delta x$ is the principal part of Δy. This we shall call the *differential* of y and denote it by dy.

The case of the independent variable x, however, is different. For in that case $f(x) = x$; therefore $y = x$, and formula (1) is simply

$$\Delta x = dx. \qquad (2)$$

There is no possibility, therefore, of separating Δx into two parts; in other words, the principal part of Δx is the whole of Δx, and we may take this as dx.

Summing up, we say:

The differential of an independent variable x is equal to the increment of the variable; that is,

$$dx = \Delta x. \qquad (3)$$

The differential of a function $y = f(x)$ is the principal part of the increment of y and is given by the formula

$$dy = f'(x)dx. \qquad (4)$$

Suppose, now, we have $y = f(x)$ and $x = \phi(t)$; then $y = F(t)$. Now, by the definition above, we have

$$dt = \Delta t,$$
$$dx = \phi'(t)dt,$$
$$dy = F'(t)dt.$$

Substituting for $F'(t)$ the value derived given in (1), § 4, we have

$$dy = f'(x)\phi'(t)dt;$$

whence

$$dy = f'(x)dx. \qquad (5)$$

Note that this is the same form as (4); but in (4) dx is the entire increment of x, whereas in (5) dx is the principal part of that increment. The result is,

The differential of a function y is given by the formula $dy = f'(x)dx$, whether x is the independent variable or not.

We are now ready to write the derivative as a quotient; namely,

$$f'(x)=\frac{dy}{dx}. \tag{6}$$

In differential form (1), § 4, becomes

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}. \tag{7}$$

15. Higher differentials. If $u=f(x)$ and $v=\phi(x)$ are two functions of x which possess derivatives, we have, by well-known formulas for differentiation,

$$\frac{d}{dx}(u+v)=f'(x)+\phi'(x),$$

$$\frac{d}{dx}(uv)=f'(x)\phi(x)+f(x)\phi'(x),$$

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{f'(x)\phi(x)-f(x)\phi'(x)}{[\phi(x)]^2};$$

whence, by the definition of the differential,

$$d(u+v)=du+dv, \tag{1}$$

$$d(uv)=v\,du+u\,dv, \tag{2}$$

$$d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}. \tag{3}$$

Let us apply formula (2) to

$$df=f'(x)dx.$$

We have $$d(df)=d[f'(x)]dx+f'(x)d(dx). \tag{4}$$

Now we have, by the definition of § 14,

$$d[f'(x)]=f''(x)dx.$$

It is natural to express $d(dy)$ by d^2y and it is customary to express $(dx)^2$ by dx^2. This must not be confused with $d(x^2)$, which, by § 14, is equal to $2\,x\,dx$.

We have, then, $$d^2f=f''(x)dx^2+f'(x)d^2x. \tag{5}$$

This is called the second differential of f.

Formula (5) contains a factor d^2x which has not been defined. However, if x is a function of another variable t, so that

$$x=F(t),$$

we have, by another application of (5), the result

$$d^2x=F''(t)dt^2+F'(t)d^2t,$$

but here d^2t is still to be defined. It is evident then that (5) is not sufficient to define the second differential of the *independent* variable. We are accordingly free to frame that definition as we will, and we say

The second differential of the independent variable is by definition zero.

With this and (5) it follows that when x is an independent variable,

$$d^2f = f''(x)dx^2; \tag{6}$$

but when x is not an independent variable,

$$d^2f = f''(x)dx^2 + f'(x)d^2x.$$

The fact that the second differential has different forms according as x is independent or not is in striking contrast to the fact that the form of the first differential is always the same. Second differentials must therefore be used with more care than the first. We notice that when x is the independent variable we have, from (6),

$$f''(x) = \frac{d^2f}{dx^2}; \tag{7}$$

whereas when x is not independent we have, from (5),

$$f''(x) = \frac{d^2f - f'(x)d^2x}{(dx)^2} = \frac{d^2f\,dx - df\,d^2x}{(dx)^3}, \tag{8}$$

which agrees with (7) only when $d^2x = 0$.

In spite of this the symbol $\frac{d^2f}{dx^2}$ is used to represent the second derivative $f''(x)$ even when x is not the independent variable. This may be explained by interpreting $\frac{d^2f}{dx^2}$ as a symbol for

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d\left(\frac{df}{dx}\right)}{dx}.$$

Then

$$\frac{d\left(\frac{df}{dx}\right)}{dx} = \frac{d[f'(x)]}{dx} = \frac{f''(x)dx}{dx} = f''(x). \tag{9}$$

It follows that $\frac{d^2f}{dx^2}$ is used in two senses: first as a symbol for the second derivative and secondly as the quotient of d^2f as given in (5) by dx^2. These two senses agree only when x is the independent variable. The context usually makes clear which sense is meant. As a matter of fact, the use of $\frac{d^2y}{dx^2}$ as a derivative is more common than the other use.

If in (8) we place $y = f(x)$, and interpret $\frac{d^2y}{dx^2}$ as a symbol for the second derivative $f''(x)$ and not as the quotient of differentials, we have

$$\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}. \tag{10}$$

This formula may also be obtained by direct differentiations, thus:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \cdot \frac{dt}{dx} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3},$$

as before.

This result may also be obtained by dividing the numerator and the denominator of the fraction in (8) by $(dt)^3$.

The third, fourth, and higher differentials are $d(d^2x) = d^3x$, $d(d^3x) = d^4x$, etc.

Since, if x is the independent variable, $d^2x = 0$, it follows at once that $d^nx = 0$. That is,

The nth differential of the independent variable is zero if n is greater than 1.

The higher differentials of a function of x are found by operating with the laws (1), (2), and (3). Thus we have

$$d^2f = f''(x)dx^2 + f'(x)d^2x. \tag{11}$$

Then
$$d^3f = f'''(x)dx^3 + 3\,f''(x)dx\,d^2x + f'(x)d^3x. \tag{12}$$

It is to be noticed that (12) gives

$$f'''(x) = \frac{d^3f}{dx^3} \tag{13}$$

only when x is the independent variable. Otherwise, if t is the independent variable we have, from (12),

$$\frac{d^3f}{dt^3} = f'''(x)\left(\frac{dx}{dt}\right)^3 + 3\,f''(x)\,\frac{dx}{dt}\frac{d^2x}{dt^2} + f'(x)\,\frac{d^3x}{dt^3}. \tag{14}$$

As in the case of the second derivative, the expression

$$\frac{d^3f}{dx^3}$$

is used for the third derivative even when x is not the independent expression. In this case

$$\frac{d^3f}{dx^3} = \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right],$$

and $\dfrac{d^3f}{dx^3}$ is not the quotient of d^3f by dx^3.

From (12) we have

$$f'''(x) = \frac{d^3f - 3\,f''(x)dx\,d^2x - f'(x)d^3x}{dx^3}.$$

By means of (8) of this section and (6), § 14, this reduces readily to the form

$$f'''(x) = \frac{dx(d^3f\,dx - df\,d^3x) - 3\,[d^2f\,dx - df\,d^2x]d^2x}{dx^5}. \tag{15}$$

If t is the independent variable, we may divide all the terms of the numerator of (15) by dt^5 and obtain a result in derivatives which may otherwise be obtained by direct differentiation.

Similar results are readily obtained for the fourth and higher differentials.

16. Change of variable. The methods and formulas just obtained may be used to solve certain problems connected with the change of the variables in a given expression.

1. Let there be given an expression involving y, $\dfrac{dy}{dx}$, and $\dfrac{d^2y}{dx^2}$, where the last symbol means a second derivative, and let it be required to replace y by z where $y = f(z)$. We have at first, by § 4,

$$\frac{dy}{dx} = f'(z)\,\frac{dz}{dx} \tag{1}$$

and then, by direct differentiation,

$$\frac{d^2y}{dx^2} = f''(z)\left(\frac{dz}{dx}\right)^2 + f'(z)\,\frac{d^2z}{dx^2}. \tag{2}$$

2. Let it be required in an expression involving y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ to replace x by t, where $x = \phi(t)$.

We have
$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{1}{\phi'(t)}\frac{dy}{dt}, \tag{3}$$

and, again by direct differentiation or by (10), § 15,

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d^2y}{dt^2}\dfrac{dx}{dt} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3} = \frac{\dfrac{d^2y}{dt^2}\phi'(t) - \dfrac{dy}{dt}\phi''(t)}{[\phi'(t)]^3}. \tag{4}$$

3. Let it be required to interchange x and y in an expression involving x, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$. This is a special case of (4) in which $t = y$, and therefore $\frac{dy}{dt} = 1$, $\frac{d^2y}{dt^2} = 0$. We have

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}, \tag{5}$$

$$\frac{d^2y}{dx^2} = -\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}. \tag{6}$$

EXERCISES

1. Prove that if $f(x)$ and $\phi(x)$ are continuous at $x = a$, then $f(x) + \phi(x)$, $f(x) \cdot \phi(x)$ are continuous at $x = a$, and that $\frac{f(x)}{\phi(x)}$ is also continuous at $x = a$ unless $\phi(a) = 0$.

2. If $y = f(x)$ is continuous when $x = a$, and $x = \phi(t)$ is continuous when $t = b$, where $a = \phi(b)$, show that $y = f[\phi(t)]$ is continuous when $t = b$.

3. Show that the theorem of the mean as given in § 6 may be translated into the theorem of the mean for definite integrals; namely,

$$\int_a^b F(x)dx = (b - a)F(\xi). \quad (a < \xi < b)$$

4. Expand e^x into Maclaurin's series and show that

$$R < \frac{x^{n+1}}{(n+1)!} \text{ if } x > 0,$$

and that
$$|R| < \frac{|x^{n+1}|}{(n+1)!} \text{ if } x < 0.$$

5. What error is made by computing $e^{\frac{1}{3}}$ by five terms of Maclaurin's series? How many terms must be taken to obtain $e^{\frac{1}{3}}$ correct to seven decimal places?

6. Show that in the expansion of $\log(1+x)$

$$|R| < \frac{x^{n+1}}{n+1} \text{ when } x > 0,$$

and that
$$|R| < \frac{|x^{n+1}|}{(n+1)(1+x)^{n+1}} \text{ when } x < 0.$$

7. From the result of Ex. 6 estimate the error made in computing log 1.2 from three terms of the series. How many terms of the series are sufficient to compute log 1.2 accurately to six decimal places?

8. From the result of Ex. 6 how many terms of the expansion of $\log(1+x)$ are sufficient to compute log .9 to five decimal places?

9. Show that in the expansion of $\log\frac{1+x}{1-x}$

$$|R| < \frac{2\,x^{n+2}}{(n+2)(1-x)^{n+2}} \text{ when } x > 0,$$

and that
$$|R| < \frac{2|x^{n+2}|}{(n+2)(1+x)^{n+2}}$$

when $x < 0$, where n is the exponent of x in the last term retained in the expansion.

10. From the result of Ex. 9 how many terms of the expansion of $\log\frac{1+x}{1-x}$ are required to compute $\log\frac{5}{3}$ to four decimal places?

11. Show that in the expansion of $(1+x)^k$

$$|R| < \frac{k(k-1)\cdots(k-n)x^{n+1}}{(n+1)!} \text{ when } x > 0,$$

and that
$$|R| < \frac{k(k-1)\cdots(k-n)}{(n+1)!\,(1+x)^{n-k+1}}|x^{n+1}| \text{ when } x < 0,$$

if $n - k + 1 > 0$.

12. From the result of Ex. 11 find how many terms of the binomial series are sufficient to compute $\sqrt{102}$ to four decimal places.

13. By integration find an expansion for $\sin^{-1} x$.

14. By division find an expansion for $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$.

15. Find an expansion for $\int_0^x e^{-x^2}dx$.

16. Find an expansion for $\int_0^x \cos x^2 dx$.

Find the limit approached by each of the following functions as the variable approaches the given value:

17. $\frac{\cos x - \cos a}{x - a}$, $x \to a$.

18. $\frac{a^x - b^x}{x}$, $x \to 0$.

19. $\frac{\log \cos 2x}{(\pi - x)^2}$, $x \to \pi$.

20. $\frac{\tan x - x}{x - \sin x}$, $x \to 0$.

21. $\frac{\sin x - x}{x - \tan x}$, $x \to 0$.

22. $\frac{\sec 3x}{\sec 5x}$, $x \to \frac{\pi}{2}$.

23. $\frac{1 - \log x}{x}$, $x \to 0$.

24. $\frac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}$, $x \to \frac{\pi}{2}$.

25. $\frac{\log x}{x^n}$, $x \to \infty$ (n positive).

26. $\frac{x^n}{e^x}$, $x \to \infty$ (n positive).

27. Find the limit approached by

$$\frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m}$$

as $x \to \infty$, where n and m are positive integers, under each of the three hypotheses $n < m$, $n = m$, $n > m$.

28. Show that for all positive and negative values of n

$$\lim_{x \to \infty} x^n e^{-x} = 0.$$

29. Show that for all positive values of n and m

$$\lim_{x \to \infty} \frac{(\log x)^m}{x^n} = 0.$$

30. Show that for all possible values of n and m

$$\lim_{x \to 0} x^m (\log x)^m = 0.$$

31. Evaluate $\lim_{x \to \pi}\left[\frac{1}{x - \pi} - \frac{1}{\sin x}\right]$.

32. Evaluate $\lim_{x \to 0} x^x$.

33. Evaluate $\lim_{x \to \infty} x^{\frac{1}{x}}$.

34. Evaluate $\lim_{x \to 0} (1 + ax)^{\frac{b}{x}}$.

35. Evaluate $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$.

36. In Fig. 16 find the order of CD.

37. Let T be the intersection of the tangents to the circle (Fig. 16) at A and B. Find the order of $TC - CD$.

38. Two sides AB and AC of a triangle differ by an infinitesimal α, and the angle θ at A is an infinitesimal of the same order as α. What order of infinitesimals is neglected if the area of the triangle is taken as $\frac{1}{2}\overline{AB}^2\theta$? as $\frac{1}{2}AB \cdot AC \cdot \theta$?

39. If $y = f(x)$ is a curve in Cartesian coördinates, show that the area bounded by the curve, the axis of x, and two ordinates separated by an infinitesimal distance Δx differs from $y\,\Delta x$ by an infinitesimal of higher order.

40. If $r = f(\theta)$ is a curve in polar coördinates, show that the area bounded by the curve and two radii making an infinitesimal angle $\Delta\theta$ with each other differs from $\frac{1}{2}r^2\Delta\theta$ by an infinitesimal of higher order.

41. What order of infinitesimals is neglected in taking as $4\pi r^2 h$ the volume of a spherical shell of finite inner radius r and infinitesimal thickness h?

42. A parallelogram has an angle which differs from 90° by an infinitesimal of first order. What order of infinitesimals is neglected by taking the area of the parallelogram as the product of two adjacent sides?

43. Show that in a hyperbola the tangent makes equal angles with the focal radii drawn to the point of contact.

44. Show that in a parabola the tangent makes equal angles with the focal radius to the point of contact and the line through the point of contact parallel to the axis.

45. A circle of radius a rolls on a straight line. A point P on its rim describes a cycloid. If P moves to P' by an infinitesimal rotation $d\phi$, show by the method of infinitesimals that $PP' = a \sin \frac{1}{2}\phi\, d\phi$. Note that the linear displacement of the circle is $a\,d\phi$, and that the motion of P takes place in a direction normal to the line from P to the point of contact of the circle with the straight line.

46. Find d^2y when $y = \sin x^2$ under the two assumptions (1) that x is the independent variable; (2) that $x = e^t$. In the latter case first use formula (5), § 15, and check by substituting for x in the given value of y.

47. In Ex. 46 find $\dfrac{d^2y}{dt^2}$ as a quotient of differentials and also by direct differentiation.

48. Given $y = e^x$, find d^3y (1) when x is the independent variable; (2) when $x = \log z$ and z is the independent variable. Verify by substitution and direct differentiation.

49. In the equation

$$(1 - y^2)\frac{d^2y}{dx^2} + y\left(\frac{dy}{dx}\right)^2 + (1 - y^2)^{\frac{3}{2}} = 0$$

place $y = \sin z$.

50. In the equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$ place $y=x^nz$.

51. In the result of Ex. 50 place $x=2\sqrt{t}$.

52. In the equation

$$(1-x^2)\frac{d^2y}{dx^2}-2\,x\frac{dy}{dx}+n(n+1)y=0$$

place $x=\cos\theta$.

53. Show that if the equation of a curve $y=f(x)$ is transformed to polar coördinates by the substitutions $x=r\cos\theta$, $y=r\sin\theta$, the derivative $\frac{d^2y}{dx^2}$ becomes

$$\frac{r^2d\theta^3+2\,d\theta\,dr^2-r\,d^2r\,d\theta}{(\cos\theta\,dr-r\sin\theta\,d\theta)^3},$$

where θ is the independent variable.

54. Show that the formula for the radius of curvature of a plane curve $y=f(x)$, namely,

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

becomes

$$\rho=\frac{[dx^2+dy^2]^{\frac{3}{2}}}{dx\,d^2y-dy\,d^2x}$$

for the curve $x=f_1(t)$, $y=f_2(t)$ where t is the independent variable.

55. Show that the formula for ρ given in Ex. 54 for the curve $y=f(x)$ becomes

$$\rho=\frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2+2\,r\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}$$

in polar coördinates.

56. Show that the formula for ρ in Ex. 54 becomes

$$\rho=\frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$$

if the curve is taken as $x=\phi(y)$.

57. If $x=r\cos\theta$, $y=r\sin\theta$, show that

$$\frac{x\frac{dy}{dx}-y}{x+y\frac{dy}{dx}}=r\frac{d\theta}{dr}.$$

Note that this is the expression for the tangent of the angle which a curve makes with a line from the origin.

CHAPTER II

POWER SERIES

17. Definitions. The expression

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \tag{1}$$

is a power series. If the number of terms is finite, the power series reduces to a polynomial; if the number of terms is unlimited, the power series is an infinite series. It is with infinite series that we are concerned in this chapter.

The series (1) is said to converge for a given value $x = x_1$ if the sum of the first n terms approaches a limit as n is indefinitely increased. The limit is called the value of the series or, somewhat inaccurately, the sum of the series for $x = x_1$.

A simple and important example of a power series is that of the geometric series

$$a_0 + a_0x + a_0x^2 + a_0x^3 + \cdots + a_0x^n + \cdots. \tag{2}$$

The sum of the first n terms of the series is, by elementary algebra,

$$a_0\frac{1-x^n}{1-x} = \frac{a_0}{1-x} - \frac{a_0x^n}{1-x}. \tag{3}$$

Now if x is numerically less than unity, the last fraction in (3) approaches zero as n increases indefinitely, and the sum of the first n terms approaches the limit

$$\frac{a_0}{1-x}.$$

Hence the geometric series (1) converges for any value of x in the interval $-1 < x < 1$.

The series (1) defines, then, the function $\dfrac{1}{1-x}$ for values of x in the interval $(-1, 1)$, but does not define the function outside of that interval.

A series which is not convergent is called divergent. As an example consider the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots.$$

Now
$$\tfrac{1}{3}+\tfrac{1}{4}>\tfrac{1}{2},$$
$$\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}>\tfrac{1}{2},$$
and in this way the sum of the first n terms may be seen to be greater than any multiple of $\frac{1}{2}$ for sufficiently large n. Hence the sum of the first n terms does not approach a limit, and the series diverges.

Return now to the general series (1) and set $x=x_1$, obtaining
$$a_0+a_1x_1+a_2x_1^2+a_3x_1^3+\cdots+a_nx_1^n+\cdots. \tag{4}$$
Let us replace each quantity by its absolute value, obtaining the new series
$$|a_0|+|a_1||x_1|+|a_2||x_1^2|+|a_3||x_1^3|+\cdots$$
$$+|a_n||x_1^n|+\cdots. \tag{5}$$

We shall prove that if (5) converges, then (4) also converges. Let s_n be the sum of the first n terms of (4), and let s_n' be the sum of the first n terms of (5). Now s_n contains a certain number of positive terms whose sum we call p_n and a certain number of negative terms whose sum we call q_n, so that
$$s_n=p_n-q_n. \tag{6}$$

The positive terms of (4) appear in (5) unchanged, whereas the negative terms of (4) appear in (5) with signs changed. Hence
$$s_n'=p_n+q_n. \tag{7}$$

Now as n increases, s_n', p_n, and q_n each increases, since each is positive. Suppose s_n' approaches a limit A. Then p_n, always increasing but always less than A, must approach a limit B,* and, similarly, q_n must approach a limit C.

Hence, from (6), s_n approaches a limit $B-C$. We have accordingly proved the proposition that if (5) converges (4) also converges.

It is to be noticed that we cannot prove conversely that if (4) converges (5) does; for s_n in (6) may approach a limit even if p_n and q_n separately do not.

Summing up, we say that *a series which converges when each term is replaced by its absolute value converges as it stands.* It is said to converge *absolutely.* The determination of the absolute convergence of a series reduces, then, to the determination of the convergence of a series of positive quantities, and for that we shall find two tests useful: the comparison test (§ 18) and the ratio test (§ 19).

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* We will assume as evident that a quantity which always increases either becomes infinite or approaches a finite limit.

18. Comparison test for convergence. *If no term of a series of positive numbers is greater than the corresponding term of a known convergent series, the first series converges. If no term of a given series is less than the corresponding term of a known divergent series of positive numbers, the first series diverges.*

Let

$$c_1 + c_2 + c_3 + \cdots + c_n + \cdots \tag{1}$$

be a series of positive numbers, and let

$$M_1 + M_2 + M_3 + \cdots + M_n + \cdots \tag{2}$$

be a known convergent series such that

$$c_n \leqq M_n \tag{3}$$

for all values of n.

Let s_n be the sum of the first n terms of (1), S_n the sum of the first n terms of (2), and M the limit of S_n.

Then, from (3), all terms M_n are positive and therefore $S_n < M$, and also from (3) $s_n \leqq S_n$, so that we have

$$s_n < M. \tag{4}$$

As n increases, s_n increases and, by (4), approaches a limit * which is either less than or equal to M.

The first part of the theorem is now proved; the second part is too obvious to need formal proof.

In applying this test it is not necessary to begin with the first term of either series, but comparison may begin with any convenient term. The terms in either series before that with which comparison begins form a polynomial the value of which is finite, and the remaining terms form the infinite series considered.

For example, consider the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots. \tag{5}$$

If $p \leqq 1$, no term of the series is less than the corresponding term of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots,$$

and therefore in this case (5) diverges.

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* See footnote on page 39.

If $p > 1$, we may compare (5) with the series

$$1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{8^p} + \cdots, \tag{6}$$

where there are four terms equal to $\frac{1}{4^p}$, eight terms equal to $\frac{1}{8^p}$, and in general 2^k terms equal to $\frac{1}{(2^k)^p}$. Now no term of (5) is greater than the corresponding term of (6), and (6) is the same as

$$1 + \frac{2}{2^p} + \frac{2^2}{(2^2)^p} + \frac{2^3}{(2^3)^p} + \cdots + \frac{2^k}{(2^k)^p} + \cdots.$$

This is a geometric series with ratio equal to

$$\frac{2}{2^p} = \left(\frac{1}{2}\right)^{p-1} < 1.$$

Hence the series (5) converges when $p > 1$ and diverges when $p \leqq 1$.

19. The ratio test for convergence. *If in a series of positive numbers the ratio of the* $(n+1)$*st term to the nth term approaches a limit* L *as* n *increases without limit, then, if* $L < 1$, *the series converges; if* $L > 1$, *the series diverges; if* $L = 1$, *the series may either converge or diverge.*

Let

$$c_1 + c_2 + c_3 + \cdots + c_n + c_{n+1} + \cdots \tag{1}$$

be a series of positive numbers, and let $\lim\limits_{n \to \infty} \frac{c_{n+1}}{c_n} = L.$

We have three cases to consider:

Case I. $L < 1$. Take r, a number such that $L < r < 1$. Then, since the ratio $\frac{c_{n+1}}{c_n}$ approaches L as a limit, this ratio must become and remain less than r for n greater than some number, say m. Then

$$c_{m+1} < rc_m,$$
$$c_{m+2} < rc_{m+1} < r^2 c_m,$$
$$c_{m+3} < rc_{m+2} < r^3 c_m,$$

and so on.

Now compare (1), beginning with the term c_m, namely

$$c_m + c_{m+1} + c_{m+2} + c_{m+3} + \cdots, \tag{2}$$

with the new series

$$c_m + rc_m + r^2 c_m + r^3 c_m + \cdots. \tag{3}$$

Each term of (2) is less than the corresponding term of (3), and (3) is a convergent series, since it is a geometric series with its ratio less than unity. Hence (2) converges by the comparison test and therefore (1) converges.

CASE II. $L > 1$. Since the ratio $\frac{c_{n+1}}{c_n}$ approaches L as a limit, this ratio eventually becomes and remains greater than unity for n sufficiently large, say $n > m$.

Then $c_{m+1} > c_m$, $c_{m+2} > c_{m+1} > c_m$, etc. Each term of (2) is then greater than the corresponding term of the divergent series

$$c_m + c_m + c_m + \cdots,$$

and therefore (1) diverges.

CASE III. $L = 1$. Neither argument given above is valid, and experience shows that the series may either converge or diverge.

As a first example consider

$$1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \cdots + \frac{n}{3^{n-1}} + \frac{n+1}{3^n} + \cdots. \tag{4}$$

The ratio of the $(n+1)$st term to the nth term is $\frac{n+1}{3n}$, which approaches the limit $\frac{1}{3}$ as n increases without limit. Hence (4) converges.

Again, consider

$$1 + \frac{2^2}{2!} + \frac{3^3}{3!} + \cdots + \frac{n^n}{n!} + \frac{(n+1)^{n+1}}{(n+1)!} + \cdots. \tag{5}$$

The ratio of the $(n+1)$st term to the nth term is

$$\frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n,$$

which approaches the limit e as n increases. Hence (5) diverges.

As a last example consider

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \frac{1}{(n+1)^p} + \cdots. \tag{6}$$

The ratio of the $(n+1)$st to the nth term is $\left(\frac{n}{n+1}\right)^p$, which approaches 1 as n is increased. Hence the ratio test fails, but it has been shown in § 18 that (6) converges if $p > 1$ and diverges if $p \leqq 1$.

20. Region of convergence. We now proceed to determine the values of x for which a power series converges. We begin with the theorem that if a power series converges for $x = x_1$, it converges absolutely for any value $x = x_2$ such that

$$|x_2| < |x_1|.$$

We assume that the series

$$a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_nx_1^n + \cdots \tag{1}$$

converges in any way and wish to prove that

$$a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_nx_2^n + \cdots \tag{2}$$

converges absolutely if $|x_2| < |x_1|$.

Since (1) converges, all its terms are finite. Hence there is a positive number M such that for all values of n

$$|a_nx_1^n| < M. \tag{3}$$

Then $$|a_nx_2^n| = |a_nx_1^n| \left|\frac{x_2}{x_1}\right|^n < M\left|\frac{x_2}{x_1}\right|^n. \tag{4}$$

Form the series

$$|a_0| + |a_1x_2| + \cdots + |a_nx_2|^n + \cdots. \tag{5}$$

By (4) each term of (5) is less than the corresponding term of the convergent geometric series

$$M + M\left|\frac{x_2}{x_1}\right| + M\left|\frac{x_2}{x_1}\right|^2 + \cdots + M\left|\frac{x_2}{x_1}\right|^n + \cdots.$$

Hence (5) converges and therefore (2) converges absolutely.

If we place $x = 0$ in the series (1) we get a_0 as the sum of the first n terms, and the limit of that sum as n increases indefinitely is still a_0. The series therefore converges for $x = 0$. This may be the only value of x for which the series converges.

If there are other values of x for which the series converges, let x_1 be such a value. Then, by the theorem just proved, the series converges absolutely for all values of x in the interval $-x_1 < x < x_1$.

Let us denote that interval on the number scale by Q_1P_1 (Fig. 22). There may be no values of x outside the interval Q_1P_1 for which the series converges. If there is such a value x_2, then x converges absolutely in a new interval $Q_2P_2 (-x_2 < x < x_2)$. If there is any value of x outside of Q_2P_2 for which the series converges, we determine a new interval Q_3P_3 for any point of which the series converges absolutely. Proceeding in this way it

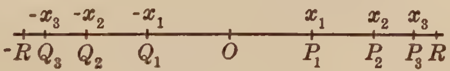

FIG. 22

is evident that the points P_1, P_2, P_3 either recede indefinitely from the origin or approach a limiting position R; that is, the series converges absolutely for all values of x or converges absolutely in a finite interval

$$-R < x < R$$

and diverges outside that interval. This interval is called the *region of convergence.* This argument does not say whether the series converges on the boundary of the region or not.

The region of convergence of the series may frequently be determined by the ratio test applied to the series of absolute values. Let us apply this test to the series (5), replacing each term by its absolute value.

We take the ratio of

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \frac{|a_{n+1}|}{|a_n|}|x| \tag{6}$$

(that this is the ratio of the $(n+2)$d term to the $(n+1)$st term instead of the ratio of the $(n+1)$st term to the nth term is unessential). Now if the ratio

$$\frac{|a_{n+1}|}{|a_n|}$$

approaches a limit L, then the ratio (6) approaches a limit $L\,|x|$ which is less than unity if $|x| < \frac{1}{L}$ and greater than unity if $|x| > \frac{1}{L}\cdot$ Hence the region of convergence is determined as

$$-\frac{1}{L} < x < \frac{1}{L}.$$

As an example take first the series

$$1 + 2\,x + 3\,x^2 + \cdots + nx^{n-1} + (n+1)x^n + \cdots.$$

The ratio of the $(n+1)$st term to the nth term is $\frac{n+1}{n}x$. The limit of this ratio as n increases indefinitely is x. The region of convergence of the series is $-1 < x < 1$.

Secondly, consider

$$1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \cdots.$$

The ratio of the $(n+1)$st term to the nth term is $\frac{x}{n}\cdot$ The limit of this ratio is 0 as n increases indefinitely no matter what the value of x. Hence the series converges for all values of x. The region of convergence is $-\infty < x < \infty$.

Finally, consider

$$1 + x + 2!\,x^2 + 3!\,x^3 + \cdots + (n-1)!\,x^{n-1} + n!\,x^n + \cdots.$$

The ratio of the $(n+1)$st term to the nth term is nx. This ratio becomes infinite with n for all values of x except $x = 0$. Hence this series converges only for $x = 0$.

21. Uniform convergence. Let $(-R, R)$ (Fig. 23) be the region of convergence of the power series

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + a_{n+1}x^{n+1} + \cdots, \qquad (1)$$

and let (a, b) be an interval lying completely in $(-R, R)$ (Fig. 23). For explicitness take $|b| > |a|$. Let the sum of the first n terms of (1) be denoted by $s_n(x)$ and the sum of the remaining terms by $r_n(x)$, where

FIG. 23

$$r_n(x) = a_nx^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots$$

and $$|r_n(x)| \leqq |a_nx^n| + |a_{n+1}x^{n+1}| + |a_{n+2}x^{n+2}| + \cdots. \qquad (2)$$

If (1) converges for x_1 it is possible to take n so great that

$$|r_n(x_1)| < \epsilon, \qquad (3)$$

where ϵ is any assigned positive quantity, but the value of n in (3) depends upon the value of x_1.

Suppose n is determined so that

$$|r_n(b)| < \epsilon.$$

Then for any value of x in the interval (a, b)

$$|x| < |b|,$$

and therefore, from (2),

$$|r_n(x)| < |r_n(b)| < \epsilon.$$

That is, it is possible to determine n so that for all values of x in the interval (a, b)

$$|r_n(x)| < \epsilon.$$

This property of a series is called *uniform convergence*, and we have shown that the power series is *uniformly convergent* in the region (a, b). The boundaries of that region may extend as closely as we please to the boundaries of the region of convergence.

22. Function defined by a power series. We shall now prove that *a power series defines a continuous function of x for values of x within the region of convergence of the series.*

Any value of x in the region of convergence determines another definite value; namely, the limit of the sum of the first n terms of the series. This limit we define as the value of the function and write

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots. \tag{1}$$

It remains to prove the continuity of $f(x)$. For that purpose let us write (1) in the form

$$f(x) = s_n(x) + r_n(x), \tag{2}$$

where $s_n(x)$ is an algebraic polynomial and $r_n(x)$ is an infinite series whose value is the difference between $s_n(x)$ and $f(x)$.

Let x_1 be a value of x in the region of convergence and consider an interval $(x_1 - \delta,\ x_1 + \delta)$ (Fig. 24). By the property of uniform convergence, if ϵ is any assigned positive quantity we can take n so great that $|r_n(x)| < \frac{\epsilon}{3}$ for all values of x in the interval $(x_1 - \delta,\ x_1 + \delta)$. This fixes n in (2).

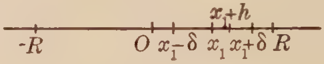

FIG. 24

Now take $x + h$ in this interval and form

$$f(x + h) = s_n(x + h) + r_n(x + h), \tag{3}$$

where

$$|r_n(x + h)| < \frac{\epsilon}{3}.$$

From (2) and (3) we have

$$f(x + h) - f(x) = s_n(x + h) - s_n(x) + r_n(x + h) - r_n(x). \tag{4}$$

Now $s_n(x)$ is an algebraic polynomial and therefore continuous. Hence h can be taken so small that

$$|s_n(x + h) - s_n(x)| < \frac{\epsilon}{3}.$$

Then in (4)

$$|f(x + h) - f(x)| \leqq |s_n(x + h) - s_n(x)| + |r_n(x + h)| + |r_n(x)|;$$

that is,

$$|f(x + h) - f(x)| < \epsilon.$$

This proves the continuity of $f(x)$.

23. Integral and derivative of a power series. We shall prove the following theorems:

I. A power series may be integrated term by term.

We mean, speaking more precisely, that if (a, b) is an interval in the region of convergence, then

$$\int_a^b f(x)dx = \int_a^b a_0\,dx + \int_a^b a_1x\,dx + \cdots + \int_a^b a_nx^n\,dx + \cdots. \tag{1}$$

To prove this, write

$$\int_a^b f(x)dx = \int_a^b s_n(x)dx + \int_a^b r_n(x)dx. \tag{2}$$

Now by the property of uniform convergence it is possible to take n so great that for all values of x in the interval (a, b)

$$|r_n(x)| < \frac{\epsilon}{b-a}.$$

Then $$\left|\int_a^b r_n(x)dx\right| < \int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon.$$

Hence in (2) $$\int_a^b f(x)dx = \operatorname*{Lim}_{n\to\infty} \int_a^b s_n(x)dx,$$

which is exactly the meaning of (1).

II. A power series may be differentiated term by term.

We mean, speaking precisely, that if

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots, \tag{3}$$

then the series $a_1 + 2\,a_2x + \cdots + na_nx^{n-1} + \cdots$ (4)

converges in the same region as (3) and represents $f'(x)$.

A simple proof of the convergence of (4) which is valid in most cases may be given by means of the ratio test (§ 19). For the limit of the ratio of the absolute values of two successive terms in (4) is the same as the limit of the ratio of two successive terms in (3), namely,

$$|x| \operatorname{Lim} \left|\frac{a_{n+1}}{a_n}\right|,$$

so that if this limit exists and is equal to L, the region of convergence of each series is

$$-\frac{1}{L} < x < \frac{1}{L}.$$

This proof fails, however, in the case in which L does not exist. Therefore we need another proof. Let x (Fig. 25) be a value of x lying in the region of convergence of (3) and let X be a quantity such that

$$x < X < R.$$

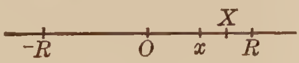

FIG. 25

Since we are dealing with absolute convergence we may for convenience take x, X, and the coefficients of (3) as positive. Taking the nth term of (4) we may write

$$na_nx^{n-1} = n\frac{a_n}{X}\left(\frac{x}{X}\right)^{n-1}X^n. \tag{5}$$

But $\dfrac{x}{X} = r$, a number less than unity, and it is easy to show by the method of undetermined forms that

$$\lim_{n \to \infty} nr^{n-1} = 0 \text{ when } r < 1.$$

Therefore n can be taken so large that

$$n\left(\frac{x}{X}\right)^{n-1} < X$$

and, from (5), $$na_n x^{n-1} < a_n X^n.$$

But the series with terms $a_n X^n$ converges, since X is in the region of convergence of the series (3); therefore the series (4) converges absolutely.

On the other hand, if x is taken outside the region of convergence and X is a value less than x, then (5) shows that eventually

$$na_n x^{n-1} > a_n X^n;$$

and since the series with terms $a_n X^n$ diverges, so does the series (4).

We have now in two ways proved that the series (4) converges in the interval $$-R < x < R.$$

Hence, by the first theorem, it may be integrated term by term, and the integral is exactly (3) plus a constant of integration. Therefore (4) is the derivative of (3).

As a consequence of this we may prove the following theorem:

III. A function may be expanded into a power series of x in only one way.

Let there be given

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

We form $$f'(x) = a_1 + 2\,a_2 x + 3\,a_3 x^2 + \cdots,$$
$$f''(x) = 2\,a_2 + 3!\,a_3 x + \cdots, \text{ etc.};$$

whence

$$a_0 = f(0),\ a_1 = f'(0),\ a_2 = \frac{1}{2!}f''(0),\ \cdots,\ a_n = \frac{1}{n!}f^n(0),$$

and we have Maclaurin's series. Since this determination of the coefficients is independent of the manner in which the original series was obtained, we have proved the theorem.

24. Taylor's series. In the foregoing sections we have considered functions defined by means of given series. Conversely, if a function is known, and known to be continuous and to possess derivatives of all orders, it may be expanded into a power series. This is an extension of the work of § 7. We saw there that if $f(x)$ and its $(n+1)$st derivatives exist and are continuous, then

$$f(x) = f(0) + f'(0)x + \cdots + f^{(n)}(0)\frac{x^n}{n!} + R, \qquad (1)$$

where $$R = \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi). \quad (0 < \xi < x)$$

Now if $f(x)$ possesses all its derivatives, and they are continuous, the formula (1) may be expanded indefinitely; and if at the same time $|R|$ approaches the limit 0, (1) becomes a convergent infinite series representing $f(x)$.

For example, consider

$$\sin x = x - \frac{x^3}{3!} + \cdots + (-1)^k\frac{x^{2k+1}}{(2k+1)!} + R, \qquad (2)$$

where $$R = \pm\frac{x^{2k+3}}{(2k+3)!}\cos\xi.$$

Hence $$|R| < \frac{|x^{2k+3}|}{(2k+3)!};$$

and whatever the value of $|x|$, the value of $|R|$ approaches zero as k increases infinitely. This is readily seen from the fact that if k is increased by unity, $|R|$ is multiplied by $\frac{|x^2|}{(2k+4)(2k+5)}$, and this factor approaches 0 as k increases. Hence (2) determines an infinite series which represents $\sin x$ for all values of x.

It should be noticed that it is not sufficient to establish the convergence of the series which would be obtained by dropping R from (1), because this series may converge but still not represent $f(x)$. In this connection, consider

$$f(x) = e^{-\frac{1}{x^2}}.$$

If the values of $f(x)$ and its derivatives for $x = 0$ are defined as equal to the limits approached as $x \rightarrow 0$, the function and its derivatives are continuous and have the value zero when $x = 0$.

Therefore Maclaurin's series is

$$e^{-\frac{1}{x^2}} = 0 + 0 + \cdots + 0 + R. \qquad (3)$$

It is manifestly absurd to omit R and say that the resulting series represents the function. In fact R is the value of the function for a given value of x and does not approach zero as a limit when the number of terms of the series is indefinitely increased.

Again, consider $\quad f(x) = \sin x + e^{-\frac{1}{x^2}}.$

We have

$$\sin x + e^{-\frac{1}{x^2}} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + R. \qquad (4)$$

If R is dropped from (4), there is left a convergent series which does not have as its limit the function on the left of the equation. In fact we have already shown that the series in question defines $\sin x$.

This example is, of course, exceptional. As a rule, if the series

$$f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

converges it represents $f(x)$ for all values of x in the region of convergence. This may be established for the elementary functions by discussing R as we have done in the case of $\sin x$. This requires obtaining the general form for the nth derivative of the function, which in many cases is difficult or even impossible to obtain. A more general proof may be given by use of the properties of the function of a complex variable (see § 147).

We have discussed the series expansion in the neighborhood of $x = 0$. To expand the function in the neighborhood of $x = x_0$ we have simply to place $x' = x - x_0$ and proceed as before for x'.

Suppose, now, that the power series

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \qquad (5)$$

defines $f(x)$ for x within a region of convergence $(-R, R)$ (Fig. 26). The series enables us to find the value of $f(x)$ and all its derivatives for $x = x_0$ when x_0 is in the region of convergence. Placing $x' = x - x_0$ we may obtain a new power series

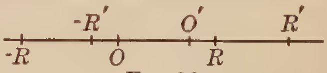

FIG. 26

$$a_0' + a_1'x' + a_2'x'^2 + a_3'x'^3 + \cdots, \qquad (6)$$

which will converge in a region $-R' < x' < R'$. If this region extends beyond the region $(-R, R)$, we have $f(x)$ defined for values of x which lie outside the region of convergence of (5). Again, taking $x'' = x' - x'_0$ we may form a series

$$a_0'' + a_1''x'' + a_2''x''^2 + \cdots,$$

which defines $f(x)$ in general for still other values of x.

In this way the values of the function defined originally by a series for certain values of x may be obtained for values which lie outside the region of convergence of the original series.

25. Operations with two power series. We shall now show that two power series may be added, multiplied together, or divided one by the other, and the result is a power series which converges when each of the first two series does and which represents the sum, product, or quotient, respectively, of the two series.

Let

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots, \tag{1}$$

$$b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + \cdots, \tag{2}$$

be two power series which converge for a given value of x, and let

$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n + \cdots \tag{3}$$

be the result obtained by adding the series term by term. We wish to show that (3) converges and represents the sum of (1) and (2).

Let s_n be the sum of the first $(n+1)$ terms of (1), s_n' the similar sum for (2), and S_n the similar sum for (3). Then

$$S_n = s_n + s_n';$$

and since s_n and s_n' approach limits, S_n approaches a limit which is the sum of s_n and s_n'.

Again, let (1) and (2) be multiplied as if they were polynomials, thus forming the series

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\ + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \cdots. \tag{4}$$

Let S_{2n} be the sum of the first $2n+1$ terms of (4), that is, of terms involving all powers of x up to and including x^{2n}, and let s_n, s_n' be as before. It is evident that all partial products formed by multiplying s_n and s_n' are found in S_{2n}, but that S_{2n} will also contain partial products not thus formed. These partial products may be arranged as follows:

$$\begin{aligned} &a_0(b_{n+1}x^{n+1} + \cdots + b_{2n}x^{2n}) \\ &+ a_1x(b_{n+1}x^{n+1} + \cdots + b_{2n-1}x^{2n-1}) \\ &+ \cdots\cdots\cdots\cdots\cdots \\ &+ a_{n-2}x^{n-2}(b_{n+1}x^{n+1} + b_{n+2}x^{n+2}) \\ &+ a_{n-1}x^{n-1}b_{n+1}x^{n+1} \\ &+ a_{n+1}x^{n+1}(b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) \\ &+ a_{n+2}x^{n+2}(b_0 + b_1x + \cdots + b_{n-2}x^{n-2}) \\ &+ \cdots\cdots\cdots\cdots\cdots \\ &+ a_{2n}x^{2n}b_0. \end{aligned} \tag{5}$$

Let the sum of the terms in (5) be represented by δ. Then

$$S_{2n} = s_n s_n' + \delta. \tag{6}$$

Since a power series converges absolutely, the absolute value of s_n is less than some finite quantity, say A, and the absolute value of s_n' is less than some finite quantity B.

The parentheses in (5) are of two types. The first type is of the form

$$b_{n+1}x^{n+1} + \cdots + b_{2n-k}x^{2n-k}. \tag{7}$$

This is less than the remainder of (2) after the $n+1$ terms have been taken in s_n'. Hence, since a power series converges absolutely, the sum (7) can be made in absolute value less than $\frac{\epsilon}{A+B}$, where ϵ is any assigned quantity, by taking n sufficiently great.

The second type of parentheses in (5) is

$$b_0 + b_1x + \cdots + b_{n-k}x^{n-k},$$

which in absolute value is less than s'_n and therefore less in absolute value than B. Hence, from (5),

$$|\delta| < (a_0 + a_1x + \cdots + a_{n-1}x^{n-1})\frac{\epsilon}{A+B} + (a_{n+1}x^{n+1} + \cdots + a_{2n}x^{2n})B. \tag{8}$$

Now, reasoning as before, the absolute value of the first parenthesis in (8) is less than A and that of the second parenthesis can be made less than $\frac{\epsilon}{A+B}$ by taking n sufficiently great. Hence it is possible so to determine n that

$$|\delta| < \epsilon.$$

Therefore, from (6), $\text{Lim } S_{2n} = \text{Lim } (s_n s_n')$.

In a similar manner it may be shown that

$$\text{Lim } S_{2n+1} = \text{Lim } (s_n s_n').$$

Hence the proposition is proved.

That one power series may be divided by another follows immediately, since division is simply a process to find a quotient which, multiplied by the divisor, gives the dividend. A more general operation is as follows:

Let
$$y = f(z) = a_0 + a_1z + a_2z^2 + \cdots \tag{9}$$
and
$$z = \phi(x) = b_0 + b_1x + b_2x^2 + \cdots. \tag{10}$$

Then
$$\begin{aligned} y = f[\phi(x)] &= F(x) \\ &= a_0 + a_1(b_0 + b_1x + b_2x^2 + \cdots) \\ &\quad + a_2(b_0 + b_1x + b_2x^2 + \cdots)^2 \\ &\quad + a_3(b_0 + b_1x + b_2x^3 + \cdots)^3 \\ &\quad + \cdots. \end{aligned} \tag{11}$$

By computing the powers of the series for z and collecting the coefficients of the powers of x we may obtain in this way an expansion of y in powers of x. In many cases the coefficients of the powers of x will be infinite number series which can be computed only approximately, but in other cases they will be exact.

For example, let us find log cos x.

We have
$$\begin{aligned} \log\cos x &= \log\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \\ &= \log(1+u) \\ &= u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots, \end{aligned}$$
where
$$u = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$
Hence
$$\begin{aligned} \log\cos x = &\left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)^2 \\ &+ \frac{1}{3}\left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)^3 - \cdots \\ = &-\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 + \cdots. \end{aligned}$$

26. The exponential and trigonometric functions. By expansion into Maclaurin's series the following series are obtained:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots. \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots. \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots. \tag{3}$$

These series converge for all real values of x and may be taken as the definitions of the functions concerned. It is true that the series have been derived by a chain of reasoning which began with elementary definitions. For example, $\sin x$ and $\cos x$ have been originally defined as ratios connected with an angle whose circular measure is x. Properties of these functions are then developed both in elementary trigonometry and in the calculus leading finally to the series above.

This process may be reversed. The series given above may now be laid down as the definition of the functions, and all the familiar properties may be deduced from them. For instance, since two series may be multiplied, it is not difficult to show from (2) and (3) that

$$\sin x \cos y + \cos x \sin y = \sin (x + y)\,;$$

and since a series may be differentiated, it follows from (2) and (3) that

$$\frac{d}{dx} \sin x = \cos x,$$

$$\frac{d}{dx} \cos x = - \sin x.$$

This method has the advantage of giving us a purely analytic definition with no dependence on a geometric construction. The connection with an angle becomes then merely an application. This definition also makes it possible to speak of the values of e^x, $\sin x$, and $\cos x$ when x has a complex imaginary value of the form $a + bi$ where $i = \sqrt{-1}$.

The study of the complex variable will be taken up in detail in Chapter XV. It is assumed now that the student has acquired from elementary algebra the knowledge that $a + bi$ may be operated on by ordinary rules of algebra, with the addition that i^2 is to be placed equal to -1. Hence if we place $x = a + bi$ in each of the series (1), (2), (3), there result two series, one not involving i, the other multiplied by i, and these define e^{a+bi}, $\sin (a + bi)$, and $\cos (a + bi)$ respectively.

However, we shall be concerned in this section only with the result of replacing x by xi in (1).

Since $\quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$, etc.,

we have

$$e^{xi} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \tag{4}$$

$$= \cos x + i \sin x.$$

Similarly, we obtain

$$e^{-xi} = \cos x - i \sin x. \tag{5}$$

From (4) and (5) follow immediately

$$\sin x = \frac{e^{xi} - e^{-xi}}{2i}, \quad \cos x = \frac{e^{xi} + e^{-xi}}{2}. \tag{6}$$

These formulas show a remarkable relation between the exponential and trigonometric functions.

By the law of multiplication of series it is easy to show from (1) the truth of the exponential law

$$e^{x_1}e^{x_2} = e^{x_1 + x_2}. \tag{7}$$

Hence we have, from (7) and (4),

$$e^{x+yi} = e^x e^{yi} = e^x(\cos y + i \sin y). \tag{8}$$

From the formula (7) it follows that

$$(e^{\theta i})^n = e^{n\theta i},$$

where n is a positive integer. Rewriting this equation in the light of (4), we have *De Moivre's theorem*; namely,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \tag{9}$$

This may also be verified for small values of n by actual multiplication and then proved by mathematical induction for n equal to any positive integer. It is also true when n is negative or fractional, but we will not go into this now.

27. Hyperbolic functions. From analogy with the sine and cosine as given in (6), § 26, we may define two new functions, called the *hyperbolic sine* and the *hyperbolic cosine*, as follows:

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2}, \\ \cosh x &= \frac{e^x + e^{-x}}{2}. \end{aligned} \tag{1}$$

Other hyperbolic functions are then defined by the equations

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x}, \\ \coth x &= \frac{1}{\tanh x}, \\ \operatorname{sech} x &= \frac{1}{\cosh x}, \\ \operatorname{cosech} x &= \frac{1}{\sinh x}. \end{aligned} \tag{2}$$

The series expansions for sinh x and cosh x are then, from (1),

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots, \tag{3}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots. \tag{4}$$

The hyperbolic sine and cosine are connected with the trigonometric sine and cosine by the relations

$$\sin ix = i \sinh x, \quad \cos ix = \cosh x, \tag{5}$$

which may be proved by replacing x by ix either in the formulas (6), § 26, or in the series (2) and (3), § 26.

It is clear that the hyperbolic functions must satisfy relations similar to those for the trigonometric functions. These may be derived directly from the definitions (1) or by substituting from (5) in the usual formulas of elementary trigonometry. We have, for example,

$$\begin{gathered}
\cosh^2 x - \sinh^2 x = 1, \\
1 - \tanh^2 x = \operatorname{sech}^2 x, \\
1 - \coth^2 x = \operatorname{cosech}^2 x, \\
\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y, \\
\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y, \\
\frac{d}{dx} \sinh x = \cosh x, \\
\frac{d}{dx} \cosh x = \sinh x, \\
\frac{d}{dx} \tanh x = \operatorname{sech}^2 x,
\end{gathered} \tag{6}$$

and others, some of which will be found in the list of exercises for the student.

The inverse hyperbolic functions are defined by the equations

$$\begin{aligned}
y &= \sinh^{-1} x \quad \text{when} \quad x = \sinh y, \\
y &= \cosh^{-1} x \quad \text{when} \quad x = \cosh y, \\
y &= \tanh^{-1} x \quad \text{when} \quad x = \tanh y,
\end{aligned} \tag{7}$$

and similar forms for the other three functions.

From the equation

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

we get

$$e^{2y} - 2\,xe^y = 1;$$

whence

$$e^y = x + \sqrt{x^2 + 1},$$

where only the positive sign of the radical is taken, since e^y is always positive. Hence

$$y = \sinh^{-1} x = \log (x + \sqrt{x^2 + 1}). \tag{8}$$

In the same manner it may be proved that

$$\cosh^{-1} x = \pm \log (x + \sqrt{x^2 - 1}), \tag{9}$$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1 + x}{1 - x}. \tag{10}$$

From the equation $x = \sinh y$

we get $dx = \cosh y \, dy;$

whence $$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}};$$

that is, $$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}, \tag{11}$$

where the plus sign is taken for the radical when x is real, since, from (1), $\cosh y$ is always positive.

Similarly, we may prove that

$$\frac{d}{dx} \cosh^{-1} x = \pm \frac{1}{\sqrt{x^2 - 1}}, \tag{12}$$

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2}. \tag{13}$$

From the last three formulas follow the formulas of integration:

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}, \tag{14}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a}, \tag{15}$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}. \tag{16}$$

28. Dominant functions. Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{1}$$

be a function defined by a power series, and let

$$\phi(x) = A_0 + A_1 x + A_2 x^2 + \cdots \tag{2}$$

be a second function defined by another power series in which the coefficients A_n are all *positive* and such that

$$A_n > |a_n|. \tag{3}$$

Then the function $\phi(x)$ is said to *dominate* the function $f(x)$ in the region of convergence of the two series.

For a given $f(x)$ there are an infinite number of dominant functions. A simple one may be found as follows:

Let r be any positive number in the region of convergence of (1). Since (1) converges absolutely for $x = r$, there is some positive number M which no term can equal or exceed in absolute value; that is,

$$|a_n| r^n < M. \tag{4}$$

Consider the function

$$\phi(x) = \frac{M}{1 - \frac{x}{r}} = M + \frac{M}{r} x + \frac{M}{r^2} x^2 + \cdots. \tag{5}$$

This series converges when $|x| < r$ and, by (4),

$$\frac{M}{r^n} > |a_n|.$$

Hence $\phi(x)$ is a dominant function in the region $(-r, r)$.

29. Conditionally convergent series. We have been concerned in the previous sections entirely with absolutely convergent series. It has been noted, however, that a series may converge even when it does not converge absolutely. Such a series is called *conditionally convergent.* For example, consider the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots. \tag{1}$$

The sum of the absolute values of the term of (1) is the harmonic series (§ 17), which is known to diverge. Hence (1) does not converge absolutely. However, it does converge as it stands. To show this let us plot on a number scale successive values of

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1} \frac{1}{n}.$$

The appearance of the graph (Fig. 27) may be compared to the swing of a pendulum, every s with an odd subscript corresponding to a swing to the right, every s with an even subscript corresponding to a swing to the left. Every swing in one direction is less than the previous swing in the other direction, since the numbers added or subtracted are growing smaller. Hence s_{2n} increases with n but remains less than 1, and therefore s_{2n} approaches a limit L.*

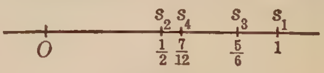

FIG. 27

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* See footnote on page 39.

Similarly, s_{2n+1} is growing smaller, but is always greater than $\frac{1}{2}$. Hence s_{2n+1} approaches a limit L'. But

$$|s_{2n+1} - s_{2n}| = \frac{1}{2n+1},$$

so that $\operatorname{Lim} |s_{n+1} - s_n| = 0$ and therefore $L = L'$.

The series (1) is a special case of a general type of series of importance for which we have the theorem

If in a series of alternately positive and negative terms each term is less in absolute value than the preceding term, and the absolute value of the terms approaches zero as a limit as the number of terms increases without limit, the series converges.

We have

$$a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^{n-1}a_n + (-1)^n a_{n+1} + \cdots, \qquad (2)$$

in which the a's are positive numbers with

$$a_{n+1} < a_n \text{ and } \operatorname*{Lim}_{n \to \infty} a_n = 0.$$

The proof that this series converges may be given on the lines on which the convergence of (1) was proved.

It should be noticed that the limit approached by the sum of n terms of a conditionally convergent series may depend upon the order in which the terms are arranged. Suppose A is any arbitrarily assumed number, which for convenience we take as positive, and let us arrange the terms as follows. Take at first just enough of the positive terms in the order in which they appear in (2) so that their sum exceeds A; then just enough of the negative terms so that the sum of all the terms taken shall be less than A; then just enough positive terms so that the sum of all the terms taken shall again be greater than A; and so on. In this way s_n is alternately greater and less than A, and it is easy to see, since the terms are decreasing in absolute value, that s_n approaches A as a limit.

The limit of the sum of n terms of an absolutely convergent series, however, does not depend upon the order in which the terms are taken.

To prove this let

$$a_0 + a_1 + a_2 + a_3 + \cdots + a_n + \cdots \qquad (3)$$

be a series of positive numbers and let

$$b_0 + b_1 + b_2 + b_3 + \cdots + b_n + \cdots \qquad (4)$$

be the same series with the terms in different order, but so that each term of (3) is found somewhere in (4) and each term of (4) is found somewhere in (3).

Let s_m' be the sum of the first m terms in (3). Then n may be taken so large in (3) that all the terms in s_m' are found in the n terms of (3). Hence

$$s_m' < s_n < A,$$

where A is the limit of the sum of the terms of (3).

Hence s_m' approaches a limit which is equal to or less than A.

In the same way the limit of s_n may be shown to be equal to or less than the limit of the sum of the terms of (4).

Hence (3) and (4) converge to the same limit.

Since an absolutely convergent series may be considered as the difference of two series of positive terms, the theorem follows.

EXERCISES

Determine the convergence or divergence of the following series:

1. $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\cdots+\frac{1}{n^2+1}+\cdots.$
2. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}+\cdots.$
3. $1+\frac{1}{2^2}+\frac{1}{3^3}+\cdots+\frac{1}{n^n}+\cdots.$
4. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\cdots.$
5. $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots+\frac{1}{(2n-1)2n}+\cdots.$
6. $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots+\frac{1}{(2n)^2}+\cdots.$
7. $1+\frac{2}{3}+\frac{2^2}{3\cdot 5}+\cdots+\frac{2^{n-1}}{3\cdot 5\cdot 7\cdots(2n-1)}+\cdots.$
8. $1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots.$
9. $\frac{3}{2}+\frac{4}{2\cdot 3}+\frac{5}{3\cdot 4}+\cdots+\frac{n+2}{n(n+1)}+\cdots.$
10. $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\cdots+\frac{1}{(4n-3)(4n-1)}+\cdots.$
11. $1+\frac{2^3}{2!}+\frac{3^3}{3!}+\cdots+\frac{n^3}{n!}+\cdots.$

12. $\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\cdots+\frac{n}{4^n}+\cdots.$

13. $\frac{3}{1\cdot 2}+\frac{3^2}{2\cdot 3}+\frac{3^3}{3\cdot 4}+\cdots+\frac{3^n}{n(n+1)}+\cdots.$

14. $1+\frac{1}{3(2!)}+\frac{1}{5(3!)}+\cdots+\frac{1}{(2n-1)n!}+\cdots.$

15. $2+\frac{2^3}{3}+\frac{2^5}{5}+\cdots+\frac{2^{2n-1}}{2n-1}+\cdots.$

16. $\frac{1}{2\cdot 1}+\frac{1}{2^3\cdot 3}+\frac{1}{2^5\cdot 5}+\cdots+\frac{1}{2^{2n-1}(2n-1)}+\cdots.$

17. $\frac{2}{1\cdot 2}+\frac{2^2}{2\cdot 3}+\frac{2^3}{3\cdot 4}+\cdots+\frac{2^n}{n(n+1)}+\cdots.$

18. $\frac{1}{2\cdot 5}+\frac{1}{3\cdot 5^2}+\frac{1}{4\cdot 5^3}+\cdots+\frac{1}{(n+1)5^n}+\cdots.$

19. $\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+\cdots+\frac{n}{3^n}+\cdots.$

20. $\frac{1}{2}+\frac{1}{5}\cdot\frac{3}{2^2}+\frac{1}{5^2}\cdot\frac{3^2}{2^3}+\cdots+\frac{1}{5^{n-1}}\cdot\frac{3^{n-1}}{2^n}+\cdots.$

Find the region of convergence of the following series:

21. $x+\frac{3}{5}x^2+\frac{4}{10}x^3+\cdots+\frac{n+1}{n^2+1}x^n+\cdots.$

22. $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^k\frac{x^{2k+1}}{(2k+1)!}+\cdots.$

23. $\frac{1}{a}+\frac{b}{a^2}x+\frac{b^2}{a^3}x^2+\cdots+\frac{b^{n-1}}{a^n}x^{n-1}+\cdots.$

24. $1+\frac{2^2}{2}x+\frac{3^2}{2^2}x^2+\cdots+\frac{n^2}{2^{n-1}}x^{n-1}+\cdots.$

25. $1+3x+\frac{3^2}{2!}x^2+\cdots+\frac{3^{n-1}}{(n-1)!}x^{n-1}+\cdots.$

26. $1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\cdots(n-k+1)}{k!}x^k+\cdots.$

27. $\frac{1}{2}+\frac{x}{2^2}+\frac{x^2}{2^3}+\cdots+\frac{x^{n-1}}{2^n}+\cdots.$

28. $\frac{x}{3}-\frac{1}{3}\frac{x^3}{3^3}+\frac{1}{5}\frac{x^5}{3^5}-\cdots+(-1)^{n-1}\frac{1}{2n-1}\frac{x^{2n-1}}{3^{2n-1}}+\cdots.$

29. $1-\frac{x^2}{1}+\frac{1\cdot 3}{1\cdot 2}x^4-\cdots+(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots 2n-3}{1\cdot 2\cdot 3\cdots n-1}x^{2n-2}+\cdots.$

30. Show that $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ converges if $\lim\limits_{n\to\infty} \sqrt[n]{a_n} < 1$ and diverges if $\lim\limits_{n\to\infty} \sqrt[n]{a_n} > 1$.

31. Show that

$$u_0 + u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots,$$

where $u_0(x)$, $u_1(x)$, $\cdots$, $u_n(x)$ are functions of (x), is uniformly convergent in a region (a, b) if a series of positive numbers

$$M_0 + M_1 + M_2 + \cdots + M_n + \cdots$$

can be found such that $|u_i(x)| < M_i$

for all values of x in the interval.

32. If $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

is an absolutely convergent series, prove that

$$a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx + \cdots$$

is uniformly convergent in the interval $(0, 2\pi)$.

33. Prove that

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^n} + \cdots$$

is not uniformly convergent in any interval which includes $x = 0$.

34. Prove that any series

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

of continuous functions $u_i(x)$ which is uniformly convergent in an interval (a, b) defines a continuous function in the interval.

35. Show that any series of continuous functions which converges uniformly in the interval (a, b) may be integrated term by term between limits which lie in that interval.

36. Show that any series of continuous functions may be differentiated term by term if the resulting series is one of continuous functions which converges uniformly.

Show by consideration of R that the following series really represent the function and determine the region of convergence:

37. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$

38. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$

39. $\log(1+x) = 1 - x + \frac{x^2}{2} - \cdots.$

40. $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots.$

41. $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots.$

Obtain expansions of the following functions by integration of an integral and determine the region of convergence:

42. $\sin^{-1} x = \int_0^x \frac{dx}{\sqrt{1-x^2}}$.
43. $\tan^{-1} x = \int_0^x \frac{dx}{1+x^2}$.
44. $\log\left(x+\sqrt{1+x^2}\right) = \int_0^x \frac{dx}{1+x^2}$.

Obtain series expansions for the following integrals and determine the region of convergence:

45. $\int_0^x e^{-x^2}\,dx$.
46. $\int_0^x \cos x^2\,dx$.
47. $\int_0^x \frac{x^{a-1}}{1+x^b}\,dx$.

Find by combinations of elementary series already obtained expansions of the following functions:

48. $\tan x = \frac{\sin x}{\cos x}$.
49. $\sec x = \frac{1}{\cos x}$.
50. $\cot x = \frac{\cos x}{\sin x}$.
51. $e^x \sec x$.
52. $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$.
53. $e^{\sin x}$.
54. $e^{\cos x} = e \cdot e^{\cos x - 1}$.
55. $e^{\tan x}$.
56 $e^{\sin^{-1} x}$.
57. $\log(1+\sin x)$.

58. Show that $\sin(x+iy) = \frac{e^y+e^{-y}}{2}\sin x + i\frac{e^y-e^{-y}}{2}\cos x$.

59. Show that

$$\cos(x+iy) = \frac{e^y+e^{-y}}{2}\cos x - i\frac{e^y-e^{-y}}{2}\sin x.$$

60. From formula (9), § 26, assuming that the real and imaginary parts of two equal complex quantities are equal, derive the formulas

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta - \cdots,$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \cdots.$$

61. Apply the method of Ex. 60 to find $\cos 3\theta$ and $\sin 3\theta$.

62. Apply the method of Ex. 60 to find $\cos 5\theta$ and $\sin 5\theta$.

63. Prove formulas (6), § 27.

64. Prove formulas (9) and (10), § 27.

65. Prove formulas (12) and (13), § 27.

66. Construct the graphs of $\sinh x$, $\cosh x$, $\tanh x$.

67. Find formulas for $\sinh 2x$ and $\cosh 2x$.

68. Find formulas for $\sinh \frac{x}{2}$ and $\cosh \frac{x}{2}$.

69. Prove that

$$\sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y,$$
$$\cosh (x + iy) = \cosh x \cos y + i \sinh x \sin y.$$

70. Show that if $a_0 = 0$ in (1), § 28, the function

$$\frac{m}{1 - \frac{x}{r}} - m$$

is a dominant function when m is any positive number whatever.

71. Show that in (1), § 28,

$$|r_n(x)| < M \frac{\left(\frac{x}{r}\right)^{n+1}}{1 - \frac{x}{r}}.$$

72. Show that the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots + (-1)^{n-1} \frac{1}{\sqrt{n}} + \cdots$$

converges conditionally but not absolutely.

73. Show that the series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

where

$$u_n = \int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x}\, dx,$$

converges.

CHAPTER III

PARTIAL DIFFERENTIATION

30. Functions of two or more variables. A quantity $f(x, y)$ is a function of two variables if the value of f is determined by the values of x and y. The values of x and y may be represented in the usual way, as coördinates of a point on a plane. Then to every point of the plane is associated a value of f, and we may speak of the value of f at a point (x, y) as a convenient way of saying the value of f for the number-pair (x, y). This manner of speaking may have a physical meaning; as, for example, if f is the intensity of illumination or electric potential at each point of the plane. The method is useful, however, when there is no physical meaning of the function and it is simply an abstract function.

The function $f(x, y)$ is continuous at a point (a, b) for which it is defined if

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = f(a, b) \tag{1}$$

independently of the manner in which x approaches a and y approaches b.

As an example of a discontinuity, consider

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}. \tag{2}$$

The function is not defined by (2) for $x = 0$, $y = 0$. It is in order, therefore, to complete the definition of $f(x, y)$ by assigning to it a definite value A for $x = 0$, $y = 0$. It is not possible, however, so to choose A that $f(x, y)$ is continuous at $(0, 0)$. For let (x, y) approach $(0, 0)$ along the straight line $y = mx$; then along this line

$$f(x, y) = \frac{1 - m^2}{1 + m^2},$$

and by changing m the function on the right may be given any value whatever. Hence $f(x, y)$ will not approach A for all ways in which $x \to 0$ and $y \to 0$. Hence $f(x, y)$ is not continuous at $(0, 0)$.

The definition of continuity may be more explicitly given as follows: Let ϵ be any assigned positive number no matter how small; then if $f(x, y)$ is continuous at (a, b), it is possible to find a number δ so that

$$|f(a+h, b+k) - f(a, b)| < \epsilon$$

for all values of h and k for which $|h| < \delta$, $|k| < \delta$.

Graphically this means that it is possible to surround (a, b) by a square of side $2\,\delta$ (Fig. 28) so that for all points in the square the difference between $f(x, y)$ and $f(a, b)$ is less than ϵ.

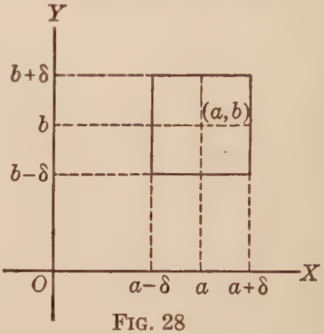

FIG. 28

A quantity $f(x, y, z)$ is a function of three variables x, y, z when its value is determined by these variables. We may interpret (x, y, z) as coördinates of a point in space and speak of the value of f at a point. Then $f(x, y, z)$ is continuous at a point (a, b, c) for which it is defined if

$$\text{Lim}\, f(x, y, z) = f(a, b, c)$$

as (x, y, z) approaches (a, b, c) in any manner whatever. More exactly: If ϵ is any small positive number, $f(x, y, z)$ is continuous at (a, b, c) if we can find another small number δ so that

$$|f(a+h, b+k, c+l) - f(a, b, c)| < \epsilon$$

for all values of h, k, and l for which $|h| < \delta$, $|k| < \delta$, $|l| < \delta$.

Geometrically this means that we may surround the point (a, b, c) with a cube of side $2\,\delta$ such that the difference between $f(x, y, z)$ and $f(a, b, c)$ for all points in the cube is less than ϵ.

Similar definitions hold for a function of four or more variables.

If f is a function of any number of variables continuous at each point of a given region, theorems I–IV of § 2 remain true, with the word "interval" interpreted as meaning a square in two dimensions, a cube in three dimensions, and so on.

31. Partial derivatives. Given $f(x, y)$ we may hold y constant and allow x to vary, thus reducing f to a function of x only, which may have a derivative defined and computed in the usual way. This derivative is called the partial derivative of f with respect to x and is denoted by the symbol of f_x or $\dfrac{\partial f}{\partial x}$ or $\left(\dfrac{df}{dx}\right)_y$. Thus, by definition,

$$f_x = \frac{\partial f}{\partial x} = \left(\frac{df}{dx}\right)_y = \text{Lim}_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}. \tag{1}$$

Again, by holding x constant we make f a function of y alone, the derivative of which is the partial derivative of f with respect to y,

$$f_y = \frac{\partial f}{\partial y} = \left(\frac{df}{dy}\right)_x = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}. \tag{2}$$

The theorem of the mean obviously holds for each of the partial derivatives; that is, using the form (3)' of § 6,

$$f(x+h, y) = f(x, y) + hf_x(x + \theta_1 h, y), \quad (0 < \theta_1 < 1) \tag{3}$$
$$f(x, y+k) = f(x, y) + kf_y(x, y + \theta_2 k), \quad (0 < \theta_2 < 1) \tag{4}$$

and, by combining these two,

$$f(x+h, y+k) = f(x, y) + hf_x(x + \theta_1 h, y) + kf_y(x + h, y + \theta_2 k). \tag{5}$$

If $f(x, y)$ has partial derivatives for each point of a domain, those derivatives are themselves functions of x and y and may have partial derivatives, which are the second partial derivatives of the function. We have then $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$; but, as we shall show presently, if $f(x, y)$ and its derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous, the order of differentiation is immaterial, so that the second partial derivatives are three in number, expressed by the symbols

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx},\\
&\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = f_{xy},\\
&\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.
\end{aligned} \tag{6}$$

Similarly, the third partial derivatives of $f(x, y)$ are four in number; namely:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3},\\
&\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\, \partial y}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^3 f}{\partial x^2\, \partial y},\\
&\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x\, \partial y}\right) = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^3 f}{\partial x\, \partial y^2},\\
&\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^3}.
\end{aligned} \tag{7}$$

So, in general, $\dfrac{\partial^{p+q}f}{\partial x^p\,\partial y^q}$ signifies the result of differentiating $f(x, y)$ p times with respect to x, and q times with respect to y, the order of differentiation being immaterial. The extension to any number of variables is obvious.

Similarly, we have partial derivatives of any orders of a function of any number of variables.

32. Order of differentiation. It remains to prove the statement that the order of differentiation is in general immaterial. We shall denote $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ by f_{yx} and $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$ by f_{xy} and shall first prove that if f has derivatives f_x, f_y, f_{yx}, and f_{xy} which are continuous at the point (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b). \tag{1}$$

For that purpose consider the expression

$$I = \frac{f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a, b)}{hk}, \tag{2}$$

and in the first place let

$$f(x+h, y) - f(x, y) = F(x, y). \tag{3}$$

Then

$$I = \frac{F(a, b+k) - F(a, b)}{hk}. \tag{4}$$

By hypothesis $f(x, y)$ is continuous and has continuous derivatives, so that $F(x, y)$ has the same properties. Hence we may apply the theorem of the mean to (4) and have

$$I = \frac{F_y(a, b+\theta_1 k)}{h} = \frac{f_y(a+h, b+\theta_1 k) - f_y(a, b+\theta_1 k)}{h}. \tag{5}$$

By our hypothesis we may apply the theorem of the mean again to (5) and have

$$I = f_{xy}(a+\theta_2 h, b+\theta_1 k). \quad (0 < \theta_1, \theta_2 < 1) \tag{6}$$

Again, let us place $f(x, y+k) - f(x, y) = \Phi(x, y)$. (7)

Then (2) gives

$$I = \frac{\Phi(a+h, b) - \Phi(a, b)}{hk}. \tag{8}$$

Applying the theorem of the mean, we have

$$I = \frac{\Phi_x(a+\theta_3 h, b)}{k} = \frac{f_x(a+\theta_3 h, b+k) - f_x(a+\theta_3 h, b)}{k}; \tag{9}$$

and, after again applying the theorem of the mean,

$$I = f_{yx}(a+\theta_3 h, b+\theta_4 k). \quad (0 < \theta_3, \theta_4 < 1) \tag{10}$$

From (6) and (10)

$$f_{xy}(a+\theta_2h,\ b+\theta_1k)=f_{yx}(a+\theta_3h,\ b+\theta_4k). \tag{11}$$

Now let $h \to 0$, $k \to 0$. Since by hypothesis f_{xy} and f_{yx} are continuous at (a, b), we have, from (11),

$$f_{xy}(a, b)=f_{yx}(a, b), \tag{12}$$

which was to be proved.

From this result the statement that the order of differentiation is immaterial for any number of differentiations or variables readily follows, of course with the proper assumptions as to continuity of the functions involved. For example, since

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

we have, by replacing f by $\dfrac{\partial f}{\partial x}$,

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right);$$

and again, interchanging the order of two differentiations,

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

so that

$$\frac{\partial^3 f}{\partial y\,\partial x^2}=\frac{\partial^3 f}{\partial x\,\partial y\,\partial x}=\frac{\partial^3 f}{\partial x^2\,\partial y}.$$

33. Differentiation of composite functions. We shall consider in this section a function of any number of variables $x, y, z, \cdots$ when $x, y, z,$ etc. are functions of other independent variables $r, s, t, \cdots$.

The number of variables in each set is immaterial. For definiteness, however, we shall consider

$$f(x, y)$$

and fix our attention on some one of the independent variables, say t. If t is given an increment Δt, then x, y, and f receive increments Δx, Δy, Δf, and

$$\begin{aligned}\Delta f &= f(x+\Delta x,\ y+\Delta y)-f(x, y)\\ &= \frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y+\epsilon_1\,\Delta x+\epsilon_2\,\Delta y,\end{aligned} \tag{1}$$

which is obtained from (5), § 31, by writing $h=\Delta x$, $k=\Delta y$ and noting that the coefficients of h and k differ by infinitesimals from f_x and f_y, respectively, under the hypothesis that these functions are continuous.

Divide (1) by Δt and take the limit as $\Delta t \to 0$.

If t is the only independent variable,

$$\operatorname*{Lim}_{\Delta t\to 0}\frac{\Delta f}{\Delta t}=\frac{df}{dt},\qquad \operatorname*{Lim}_{\Delta t\to 0}\frac{\Delta x}{\Delta t}=\frac{dx}{dt},\text{ etc.,}$$

and we have

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{2}$$

If, however, there are other independent variables besides t,

$$\operatorname*{Lim}_{\Delta t\to 0}\frac{\Delta f}{\Delta t}=\frac{\partial f}{\partial t},\qquad \operatorname*{Lim}_{\Delta t\to 0}\frac{\Delta x}{\Delta t}=\frac{\partial x}{\partial t},\text{ etc.,}$$

and we have

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}. \tag{3}$$

If we differentiate (2) with respect to t, we have

$$\frac{d^2f}{dt^2}=\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt}+\frac{\partial f}{\partial x}\frac{d^2x}{dt^2}+\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt}+\frac{\partial f}{\partial y}\frac{d^2y}{dt^2}. \tag{4}$$

Now $\frac{\partial f}{\partial x}$ is a function of x and y, and therefore (2) may be applied, with f replaced by $\frac{\partial f}{\partial x}\cdot$ Then we have

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt}+\frac{\partial^2 f}{\partial x\,\partial y}\frac{dy}{dt}.$$

Similarly,

$$\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial y}\frac{dx}{dt}+\frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}.$$

Substituting in (4), we have

$$\frac{d^2f}{dt^2}=\frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2+2\frac{\partial^2 f}{\partial x\,\partial y}\frac{dx}{dt}\frac{dy}{dt}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2+\frac{\partial f}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial f}{\partial y}\frac{d^2y}{dt^2}. \tag{5}$$

Similarly, starting with (3) we get, when x and y depend upon s and t,

$$\frac{\partial^2 f}{\partial t^2}=\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2+2\frac{\partial^2 f}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t}+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2+\frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial t^2}+\frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial t^2} \tag{6}$$

and

$$\frac{\partial^2 f}{\partial s\,\partial t}=\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t}+\frac{\partial^2 f}{\partial x\,\partial y}\left(\frac{\partial x}{\partial s}\frac{\partial y}{\partial t}+\frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right)+\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t}$$
$$+\frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t}+\frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t}. \tag{7}$$

Obviously (7) reduces to (6) if $s=t$.

In a similar manner expressions for the third and higher derivatives may be found.

We shall consider special cases of importance with changes in notation.

CASE I. $f(u)$, where $u = \phi(x)$.

Here formulas (2) and (5) take the following forms:

$$\frac{df}{dx} = f'(u)\frac{du}{dx},$$

$$\frac{d^2f}{dx^2} = f''(u)\left(\frac{du}{dx}\right)^2 + f'(u)\frac{d^2u}{dx^2}.$$

CASE II. $f(u)$, where $u = \phi(x, y)$.

We have, from (3), (5), (6), and (7),

$$\frac{\partial f}{\partial x} = f'(u)\frac{\partial u}{\partial x}, \quad \frac{\partial f}{\partial y} = f'(u)\frac{\partial u}{\partial y},$$

$$\frac{\partial^2 f}{\partial x^2} = f''(u)\left(\frac{\partial u}{\partial x}\right)^2 + f'(u)\frac{\partial^2 u}{\partial x^2},$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = f''(u)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + f'(u)\frac{\partial^2 u}{\partial x\,\partial y},$$

$$\frac{\partial^2 f}{\partial y^2} = f''(u)\left(\frac{\partial u}{\partial y}\right)^2 + f'(u)\frac{\partial^2 u}{\partial y^2}.$$

As an example, let it be required to show that $f(x^2 + y^2)$ satisfies the equation $y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} = 0$. We place $u = x^2 + y^2$. Then $f(x^2 + y^2) = f(u)$ and, from the equations given above,

$$\frac{\partial f}{\partial x} = 2\,xf'(u), \quad \frac{\partial f}{\partial y} = 2\,yf'(u);$$

whence the required result follows.

CASE III. $f(u, v)$, where $u = \phi_1(x)$, $v = \phi_2(x)$.

Formulas (2) and (4) may be written with a change of x to u, y to v, and t to x.

CASE IV. $f(u, v)$, where $u = \phi(x, y)$, $v = \psi(x, y)$.

Formula (3) with a slight change of notation gives us $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, formula (6) gives $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$, and formula (7) gives $\frac{\partial^2 f}{\partial x\,\partial y}$.

As an example, consider $f(x-y,\ y-x)$ and place $u=x-y$, $v=y-x$. Hence, from (2),

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v},$$
$$\frac{\partial f}{\partial y}=-\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v},$$

from which it appears that $f(x-y,\ y-x)$, no matter what the form of f, satisfies the equation

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=0.$$

Again, consider $z=f_1(x+at)+f_2(x-at)$ and place $u=x+at$, $v=x-at$. Applying (6) we have

$$\frac{\partial^2 z}{\partial x^2}=\frac{d^2 f_1}{du^2}+\frac{d^2 f_2}{dv^2},$$
$$\frac{\partial^2 t}{\partial t^2}=a^2\frac{d^2 f_1}{du^2}+a^2\frac{d^2 f_2}{dv^2};$$

whence it appears that z satisfies the equation

$$\frac{\partial^2 t}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2}.$$

CASE V. An important application of Case IV occurs when we have a function $f(x, y)$ in which we wish to replace x and y by polar coördinates r and θ, where

$$x=r\cos\theta,\quad y=r\sin\theta. \tag{8}$$

Then f becomes a function of r and θ and, by (3),

$$\begin{aligned}\frac{\partial f}{\partial r}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta,\\ \frac{\partial f}{\partial \theta}&=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}=-r\frac{\partial f}{\partial x}\sin\theta+r\frac{\partial f}{\partial y}\cos\theta.\end{aligned} \tag{9}$$

By solving these equations we obtain

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial f}{\partial r}\cos\theta-\frac{\partial f}{\partial \theta}\frac{\sin\theta}{r},\\ \frac{\partial f}{\partial y}&=\frac{\partial f}{\partial r}\sin\theta+\frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}.\end{aligned} \tag{10}$$

These last equations may also be obtained directly from (3), written in the form

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}=\frac{\partial f}{\partial r}\frac{x}{\sqrt{x^2+y^2}}-\frac{\partial f}{\partial \theta}\frac{y}{x^2+y^2},\\ \frac{\partial f}{\partial y}&=\frac{\partial f}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}=\frac{\partial f}{\partial r}\frac{y}{\sqrt{x^2+y^2}}+\frac{\partial f}{\partial \theta}\frac{x}{x^2+y^2},\end{aligned}\tag{11}$$

where to get $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y},$

we write $r=\sqrt{x^2+y^2}, \quad \theta=\tan^{-1}\frac{y}{x}.$

Again, by use of (6), we may get

$$\frac{\partial^2 f}{\partial r^2}=\frac{\partial^2 f}{\partial x^2}\cos^2\theta+2\frac{\partial^2 f}{\partial x\,\partial y}\cos\theta\sin\theta+\frac{\partial^2 f}{\partial y^2}\sin^2\theta,$$

$$\begin{aligned}\frac{\partial^2 f}{\partial \theta^2}=r^2\frac{\partial^2 f}{\partial x^2}\sin^2\theta-2r^2\frac{\partial^2 f}{\partial x\,\partial y}\cos\theta\sin\theta+r^2\frac{\partial^2 f}{\partial y^2}\cos^2\theta\\ -r\frac{\partial f}{\partial x}\cos\theta-r\frac{\partial f}{\partial y}\sin\theta,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial r^2}\frac{x^2}{x^2+y^2}-2\frac{\partial^2 f}{\partial r\,\partial \theta}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}+\frac{\partial^2 f}{\partial \theta^2}\frac{y^2}{(x^2+y^2)^2}\\ +\frac{\partial f}{\partial r}\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}+2\frac{\partial f}{\partial \theta}\frac{xy}{(x^2+y^2)^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial r^2}\frac{y^2}{x^2+y^2}+2\frac{\partial^2 f}{\partial r\,\partial \theta}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}+\frac{\partial^2 f}{\partial \theta^2}\frac{x^2}{(x^2+y^2)^2}\\ +\frac{\partial f}{\partial r}\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}-2\frac{\partial f}{\partial \theta}\frac{xy}{(x^2+y^2)^2}.\end{aligned}$$

In this way we verify the important relation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\frac{1}{r}\frac{\partial f}{\partial r}.\tag{12}$$

34. Euler's theorem on homogeneous functions. A function $f(x, y)$ is said to be homogeneous of the nth degree in x and y if the multiplication of x and y each by a factor λ multiplies the function by λ^n; if, symbolically, $f(\lambda x, \lambda y)=\lambda^n f(x, y)$. (1)

Thus $x^2-3xy+4y^2$ is homogeneous of the second degree, $\sqrt{x^2+y^2}$ is homogeneous of the first degree, and $\frac{x-3y}{x+y}$, $e^{\frac{y}{x}}$ are each homogeneous of the zero-th degree.

Euler's theorem in its simplest form is as follows: *If $f(x, y)$ is a homogeneous function of the nth degree, then*

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf. \tag{2}$$

To prove this differentiate (1) with respect to λ. We have, placing $u=\lambda x$, $v=\lambda y$, and applying (2), § 33,

$$x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}=n\lambda^{n-1}f. \tag{3}$$

This is true for all values of λ. It is therefore true when $\lambda=1$. Substituting $\lambda=1$ in (3) gives (2), which was to be proved.

Differentiate (3) again with respect to λ. We have

$$\left(x\frac{\partial}{\partial u}+y\frac{\partial}{\partial v}\right)^2 f=n(n-1)\lambda^{n-2}f,$$

and, placing $\lambda=1$, $\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f=n(n-1)f.$ (4)

So by successive differentiations we have

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^k f=n(n-1)\cdots(n-k+1)f. \tag{5}$$

These results may obviously be obtained for any number of variables, so that we have as the general form of Euler's theorem

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\cdots\right)^k f=n(n-1)\cdots(n-k+1)f. \tag{6}$$

35. Directional derivative. Let P (Fig. 29) be any point of the plane at which $f(x, y)$ is defined and has partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Draw any curve PS through P, take Q any point on PS near to P, and let $PQ=\Delta s$, where s is the length of the curve. Let Δx and Δy be the increments of x and y corresponding to Δs, and let Δf be the change in f as (x, y) moves from P to Q. Then $\lim\limits_{\Delta s\to 0}\frac{\Delta f}{\Delta s}$ is the rate of change of f along the curve PS. Also

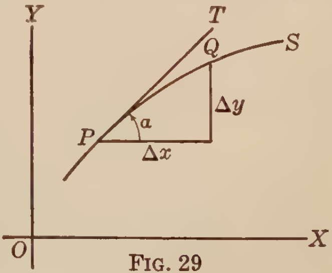

FIG. 29

$$\lim_{\Delta s\to 0}\frac{\Delta x}{\Delta s}=\frac{dx}{ds}=\cos\alpha,\qquad \lim_{\Delta s\to 0}\frac{\Delta y}{\Delta s}=\frac{dy}{ds}=\sin\alpha,$$

where α is the angle which PT, the tangent to PS at P, makes with a line parallel to OX. But from (2), § 33,

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha. \tag{1}$$

It appears that if the point P is fixed and therefore $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are fixed, the value of $\frac{df}{ds}$ depends on the angle α and not on the equation of the curve PS. Hence (1) is called the *directional derivative* of f in the direction α.

As α changes, P being fixed, the directional derivative changes. Therefore we may write

$$\frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha = F(\alpha).$$

When $\alpha = 0$ we have
$$F(0) = \frac{\partial f}{\partial x},$$

and when $\alpha = \frac{\pi}{2}$ we have
$$F\left(\frac{\pi}{2}\right) = \frac{\partial f}{\partial y}.$$

That is, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are the rates of change of f parallel to OX and OY respectively.

The directional derivative is zero when

$$F(\alpha) = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha = 0;$$

that is, when
$$\tan\alpha = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \tag{2}$$

This determines two values of α, α_1 and $\alpha_1 + \pi$, differing by 180°. Between these two values there must be, by Rolle's theorem, at least one value of α, say α_2, for which $F(\alpha)$ has its maximum value and

$$F'(\alpha_2) = -\frac{\partial f}{\partial x}\sin\alpha_2 + \frac{\partial f}{\partial y}\cos\alpha_2 = 0;$$

whence
$$\tan\alpha_2 = \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}. \tag{3}$$

An easy substitution from (3) in (1) gives

$$F(\alpha_2) = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}. \tag{4}$$

From (2) and (3) it follows that

$$\tan\alpha_1 \tan\alpha_2 = -1,$$

and therefore the directions α_1 and α_2 are at right angles to each other.

We may interpret these results geometrically still further. Let us imagine that we have marked on the plane of (x, y) those points for which $f(x, y)$ has a constant value c. By connecting such points we have a plot of the equation

$$f(x, y) = c$$

which will be in general a continuous curve.

Let the same thing be done for the equations

$$f(x, y) = c_2, \qquad f(x, y) = c_3.$$

The plane is then covered with a set of curves called *contour lines*, along any one of which $f(x, y)$ has a constant value. Thus, for $f(x, y) = x^2 + y^2$ the contour lines are concentric circles $x^2 + y^2 = c$, and for $f(x, y) = xy$ the contour lines are the equilateral hyperbolas $xy = c$. The expression "contour lines" is borrowed from a topographical map, where such lines give the projections on the plane of the map of points at which the height above sea level is constant. Along a contour line the rate of change of f is zero. Hence the angle α_1 determined by (2) gives the direction of the contour line.

Suppose now we take two contour lines, and let the difference between the values of f on the two lines be Δf. Draw a curve PN (Fig. 30) perpendicular to both lines and any curve PS making an angle $\phi = SPN$ with PN. Let $PN = \Delta n$ and let $PS = \Delta s$.

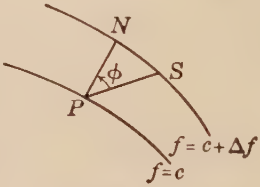

FIG. 30

Then, by § 13, $\qquad \Delta n = \Delta s \cos\phi + \epsilon,$

where ϵ is an infinitesimal of higher order than Δs. Now Δf is the same, whether taken along PN or PS. Hence

$$\frac{\Delta f}{\Delta s} = \frac{\Delta f}{\Delta n}\cos\phi + \epsilon_1,$$

and in the limit

$$\frac{df}{ds} = \frac{df}{dn}\cos\phi. \tag{5}$$

This shows that $\frac{df}{ds} < \frac{df}{dn}$, and therefore $\frac{df}{dn}$ is the maximum rate of change of f at the point P. Therefore the direction of PN is the angle α_2 given in (3).

As special cases of (5)

$$\frac{\partial f}{\partial x} = \frac{df}{dn} \cos \alpha, \tag{6}$$

$$\frac{\partial f}{\partial y} = \frac{df}{dn} \sin \alpha, \tag{7}$$

and therefore

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{df}{dn}\right)^2, \tag{8}$$

which agrees with (4). The quantity $\frac{df}{dn}$ is called the *gradient* of f. To sum up:

The gradient of a function of x and y is the maximum rate of change of the function at a point. It takes place in a direction normal to a contour line and is equal to $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$.

The extension to a function of three variables $f(x, y, z)$ is obvious. We construct in space the contour surfaces

$$f(x, y, z) = c,$$

along each of which f is constant. We may use Fig. 30, interpreting PN and PS as drawn between two such surfaces, PN being normal to both. Then, as before,

$$\frac{df}{ds} = \frac{df}{dn} \cos \phi, \tag{9}$$

showing that $\frac{df}{dn}$ is the maximum rate of change. Special cases of (9) are

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{df}{dn} \cos \alpha, \\ \frac{\partial f}{\partial y} &= \frac{df}{dn} \cos \beta, \\ \frac{\partial f}{\partial z} &= \frac{df}{dn} \cos \gamma, \end{aligned} \tag{10}$$

where α, β, γ are the angles made by PN with the axes of x, y, and z respectively. Then

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{df}{dn}\right)^2, \tag{11}$$

since

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.^*$$

---

[*] The student to whom this relation is not familiar will find a demonstration in § 45.

The gradient of a function of three variables is the maximum rate of change of the function at a point. It takes place in a direction perpendicular to the contour surfaces and is equal to

$$\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}.$$

36. The first differential. Considering a function of two variables $f(x, y)$, let x take an increment $\Delta x = h$ and y an increment $\Delta y = k$. Then f takes an increment Δf, where

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \tag{1}$$

as has been shown in § 33.

The third term here is in general an infinitesimal of lower order than the first term, and the fourth term is in general of lower order than the second, but there is no means of comparing Δx and Δy as infinitesimals.

However, we shall take the first two terms of (1) and call them the differential of f, writing

$$df = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y. \tag{2}$$

We now complete the definition of df by saying that if *x and y are independent variables,*

$$dx = \Delta x, \qquad dy = \Delta y. \tag{3}$$

Then (2) takes the form

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{4}$$

Expression (4) is called the total differential of $f(x, y)$.

The methods and results of this section may obviously be extended to functions of any number of variables. For example, for $f(x, y, z)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz. \tag{5}$$

This definition has been based upon the assumption that x and y are independent variables. We need to examine the cases in which this is not true.

We consider first the case in which

$$x = \phi_1(t), \quad y = \phi_2(t),$$

so that

$$f(x, y) = F(t).$$

Here t is the independent variable. Hence, by § 14,

$$dt = \Delta t, \quad dx = \phi_1'(t)dt, \quad dy = \phi_2'(t)dt, \quad dF = F'(t)dt. \tag{6}$$

But, by (2), § 33, $F'(t) = \frac{\partial f}{\partial x}\phi_1'(t) + \frac{\partial f}{\partial y}\phi_2'(t)$.

Multiplying through by dt and applying (6), we have

$$dF = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy,$$

which is the same form as before.

Again, let us suppose that x and y are functions of three independent variables u, v, w. Then f is a function of these same variables. From (5), since u, v, and w are independent,

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw, \tag{7}$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw, \tag{8}$$

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw. \tag{9}$$

But, from (3), § 33,
$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u},$$
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v},$$
$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w}.$$

Substituting in (9), we have

$$df = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw\right);$$

whence, by (7) and (8), $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$,

again the same result as before.

It is clear that the results obtained are independent of the number of variables involved, and we have the following theorem:

I. The differential of a function $f(x, y, z, \cdots)$ is always

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \cdots,$$

whether $x, y, z, \cdots$ are independent variables or not.

We shall next prove the following theorem:

II. If $f(x, y, z, \cdots) = c$, where c is a constant, then $df = 0$.

The relation $f(x, y, z, \cdots) = c$ cannot exist when $x, y, z, \cdots$ are independent variables unless it is an identity. We will therefore suppose that $x, y, z, \cdots$ are functions of independent variables $u, v, w, \cdots$. By substitution f becomes a function of $u, v, w, \cdots$, which is necessarily an identity, and, by I,

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw + \cdots.$$

But since $u, v, w, \cdots$ are independent variables, u may be changed without changing any of the others or the value of f. Therefore

$$f(u, v, w, \cdots) = c,$$

$$f(u + \Delta u, v, w, \cdots) = c,$$

and

$$f(u + \Delta u, v, w, \cdots) - f(u, v, w, \cdots) = 0;$$

whence

$$\frac{\partial f}{\partial u} = \operatorname*{Lim}_{\Delta u \to 0} \frac{f(u + \Delta u, v, w, \cdots) - f(u, v, w, \cdots)}{\Delta u} = 0.$$

In a similar manner $\dfrac{\partial f}{\partial v} = 0, \quad \dfrac{\partial f}{\partial w} = 0, \quad \cdots,$

and hence $df = 0$. Therefore, by I,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \cdots = 0.$$

It is important that the student should understand just what is meant. Consider, for example,

$$f(x, y, z) = x^2 + y^2 + z^2 = a^2,$$

which defines a sphere in space. Here x, y, and z cannot be independent variables, and evidently

$$\frac{\partial f}{\partial x} = 2x \neq 0, \quad \frac{\partial f}{\partial y} = 2y \neq 0, \quad \frac{\partial f}{\partial z} = 2z \neq 0.$$

But x, y, z can be expressed in terms of two independent variables u and v, where

$$x = a\cos u \sin v, \quad y = a \sin u \sin v, \quad z = a \cos v.$$

Then

$$f(x, y, z) = (a \cos u \sin v)^2 + (a \sin u \sin v)^2 + (a \cos v)^2 = a^2$$

is an identity, and $\dfrac{\partial f}{\partial u} = 0, \quad \dfrac{\partial f}{\partial v} = 0.$

Hence
$$\frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv = 0$$
for all values of du and dv, and
$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$$
for all values of dx, dy, dz which are consistent with the given equation.

III. If by any means we have found that
$$df = X_1\,dx_1 + X_2\,dx_2 + \cdots + X_n\,dx_n,$$
then $X_i = \frac{\partial f}{\partial x_i}$, *where the partial derivative is to be taken under the assumption that all variables except* x_i *are constant.*

The proof is simple. If all variables except x_i are constant, all differentials $dx_k = 0$ except for $k = i$, and the above equation becomes
$$df = X_i\,dx_i;$$
whence
$$X_i = \frac{df}{dx_i} = \frac{\partial f}{\partial x_i}.$$

It does not follow, however, that any expression such as
$$X_1\,dx_1 + X_2\,dx_2 + \cdots + X_n\,dx_n$$
which may be written down is equal to the differential of some function. In this connection we will prove the following theorem:

IV. The necessary and sufficient condition that an expression
$$M\,dx + N\,dy,$$
where M and N are functions of x and y, should be the exact differential of some function f(x, y) is that M and N should satisfy the equation
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{10}$$

Suppose, first, that $M\,dx + N\,dy$ is an exact differential, so that we have
$$df = M\,dx + N\,dy.$$
Then
$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y},$$
and
$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x},$$
so that condition (10) is necessary.

Again, let us assume that (10) is satisfied and we wish to show that a function $f(x, y)$ may be found so that

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N.$$

To do this we may first integrate $M\,dx$, regard y as constant, and have

$$\int M\,dx = \phi(x, y).$$

We now write
$$f(x, y) = \phi(x, y) + \psi(y), \tag{11}$$
where $\psi(y)$ is a function of y only, and we shall show that it is possible to determine ψ so that f is the function required. From (11),

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = \frac{\partial \phi}{\partial y} + \psi'(y),$$

which must equal N.

Hence
$$\psi'(y) = N - \frac{\partial \phi}{\partial y}. \tag{12}$$

This equation will be absurd unless the right side is free of x, since the left side is free of x by hypothesis. The condition for this is that the partial derivative with respect to x should be zero; that is, that

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\int M\,dx\right] = \frac{\partial M}{\partial y};$$

which is exactly (10). Then (12) will determine $\psi(y)$ and (11) will determine f. Hence the condition (10) is sufficient.

Extending this to three variables, we have the theorem:

V. The necessary and sufficient condition that

$$P\,dx + Q\,dy + R\,dz$$

should be an exact differential is that P, Q, and R satisfy the equations

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}. \tag{13}$$

In the first place, if

$$P\,dx + Q\,dy + R\,dz = df,$$

then
$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}, \quad R = \frac{\partial f}{\partial z};$$

whence equations (13) immediately follow.

Again, let the conditions (13) be satisfied. Then, by theorem IV,

$$P\,dx + Q\,dy = d\phi,$$

where ϕ is a function of x, y, z and z is to be regarded as constant. We form the function

$$f(x, y, z) = \phi(x, y, z) + \psi(z) \tag{14}$$

and shall show that ψ can be determined so that f is the required function. From (14), $\frac{\partial f}{\partial x} = P$, $\frac{\partial f}{\partial y} = Q$, and we must have

$$\frac{\partial f}{\partial z} = \frac{\partial \phi}{\partial z} + \psi'(z) = R,$$

so that

$$\psi'(z) = R - \frac{\partial \phi}{\partial z}. \tag{15}$$

This equation is a contradiction unless

$$\frac{\partial R}{\partial x} - \frac{\partial^2 \phi}{\partial x\,\partial z} = \frac{\partial R}{\partial x} - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 0,$$

$$\frac{\partial R}{\partial y} - \frac{\partial^2 \phi}{\partial y\,\partial z} = \frac{\partial R}{\partial y} - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0.$$

But these are just the conditions (13). Hence (15) determines $\psi(z)$ and (14) gives the required f.

It is to be noticed that it is possible to have in applied mathematics expressions of the type

$$dw = M\,du + N\,dv,$$

expressing relations between infinitesimal increments dw, du, and dv, where M and N do not satisfy the condition (10). Then w is not a function of u and v, the partial derivatives of which are M and N. For example, if H is the heat, U the energy, p the pressure, and v the volume of a gas, a small amount of heat dH added to the gas causes an increase dU in the energy and does an amount of work $p\,dv$ in expanding the gas, and we have

$$dH = dU + p\,dv.$$

The right-hand side of this equation does not satisfy (10), since p is not constant. Therefore H is not a function of v and U in the sense that it is determined when v and U are given. In fact, the amount of heat in a gas depends not merely on its present state but on the manner in which that state has been reached.

Again, suppose a force with components X and Y applied to a particle. Denote by dW the infinitesimal amount of work done by displacing the particle from (x, y) to $(x + dx, y + dy)$. It is easily shown that

$$dW = X\,dx + Y\,dy,$$

but it does not follow that W is a function of x and y. In fact, that this may be so it is necessary and sufficient that the force should be such that

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}.$$

These equations are satisfied by many of the common forces. Such forces are called conservative. But other forces, of which friction is an example, do not satisfy the condition.

These statements do not contradict theorem III, since it is there assumed that f is a function known to exist.

37. Higher differentials. We have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy; \tag{1}$$

whence

$$d(df) = d\left(\frac{\partial f}{\partial x}\right)dx + \frac{\partial f}{\partial x}\,d^2x + d\left(\frac{\partial f}{\partial y}\right)dy + \frac{\partial f}{\partial y}\,d^2y. \tag{2}$$

From (1), replacing f by $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$, we have

$$d\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}\,dx + \frac{\partial^2 f}{\partial x\,\partial y}\,dy,$$

$$d\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}\,dx + \frac{\partial^2 f}{\partial y^2}\,dy.$$

Substituting these values in (2), we have

$$d^2f = \frac{\partial^2 f}{\partial x^2}\,dx^2 + 2\,\frac{\partial^2 f}{\partial x\,\partial y}\,dx\,dy + \frac{\partial^2 f}{\partial y^2}\,dy^2 + \frac{\partial f}{\partial x}\,d^2x + \frac{\partial f}{\partial y}\,d^2y. \tag{3}$$

Equation (3) does not define d^2x and d^2y. If x and y are functions of other variables, say r and s, then an equation similar to (3) would give d^2x and d^2y but leave d^2r and d^2s undetermined. It is therefore necessary to define the differentials of the independent variables, which we shall do as in § 15; namely,

All differentials of the independent variables higher than the first are taken as zero.

Hence in (3), if x and y are independent variables, the last two terms disappear; but if x and y are functions of other variables, these terms must be retained.

Expressions for $d^n f$ $(n>2)$ are readily formed in the same way.

38. Taylor's series. We wish to extend the results of § 7 to $f(x, y)$ under the assumption that at (a, b) the function $f(x, y)$ is continuous and has continuous partial derivatives. Let us place

$$x = a + ht,\ y = b + kt,$$

where a, b, h, and k are constant and t is a variable. Then

$$f(x, y) = f(a + ht, b + kt) = F(t). \tag{1}$$

Then, by Maclaurin's theorem,

$$f(x, y) = F(0) + F'(0)t + F''(0)\frac{t^2}{2!} + \cdots + F^{(n)}(0)\frac{t^n}{n!} + R. \tag{2}$$

By (2) and (5) of § 33 we have

$$F'(t) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f,$$

$$F''(t) = h^2\frac{\partial^2 f}{\partial x^2} + 2\,hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f,$$

where the meaning of the symbols on the right of these equations is obvious. Similarly,

$$\begin{aligned} F'''(t) &= h^3\frac{\partial^3 f}{\partial x^3} + 3\,h^2k\frac{\partial^3 f}{\partial x^2\,\partial y} + 3\,hk^2\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3} \\ &= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f. \end{aligned}$$

So, in general,
$$F^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f. \tag{3}$$

This may be proved by showing by direct differentiation that if (3) is true for any value $n = k$, it is true for $n = k + 1$. Then, since (3) is certainly true for $n = 3$, it is always true.

Therefore
$$F^{(n)}(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)_0^n f,$$

where the subscript indicates that the values $t = 0$, or $x = a$, $y = b$, are to be substituted after differentiating.

Substituting in (2), we may write

$$F(t) = F(0) + \left(ht\frac{\partial}{\partial x} + kt\frac{\partial}{\partial y}\right)_0 f + \cdots + \frac{1}{n!}\left(ht\frac{\partial}{\partial x} + kt\frac{\partial}{\partial y}\right)_0^n f + R.$$

This is true for all values of t. Place $t = 1$. Then, from (1) and the last result, we have

$$f(x, y) = f(a + h, b + k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)_0 f$$
$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)_0^2 f + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)_0^n f + R. \quad (4)$$

This result may be written in another form by placing $h = x - a$, $k = y - b$.

It is easy to show from § 7 that

$$R = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)_{(\xi, \eta)}^{n+1} f, \quad (5)$$

where the subscripts indicate that we are to substitute $x = \xi$, $y = \eta$ where $a < \xi < a + h$, $b < \eta < b + k$.

In a similar manner we may show that

$$f(x, y, z) = f(a + h, b + k, c + l)$$
$$= f(a, b, c) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)_0 f + \cdots. \quad (6)$$

We have seen that a given function $f(x, y)$ may be expanded into a Taylor's series with remainder R. If all the derivatives of $f(x, y)$ exist for a given point (x, y) and if $R \rightarrow 0$ as n increases without limit, the series becomes an infinite series representing $f(x, y)$.

Conversely, a series

$$a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + \cdots \quad (7)$$

defines a function $f(x, y)$ for all values of x and y for which it converges. The definitions of convergence and absolute convergence as given in § 20 stand. It is not difficult to show by methods analogous to those in § 20 that if (1) converges for $x = x_1$, $y = y_1$, it converges absolutely for any value of x, y such that $-R < x < R$, $-S < y < S$. The region of convergence is then a rectangle in the (x, y) plane such that for any point within it $-R < x < R$, $-S < y < S$. It is also possible to show that the series converges uniformly inside any rectangle which lies in the region of convergence and hence defines a continuous function of x and y. Also, the partial derivatives of $f(x, y)$ may be obtained by differentiating the series term by term.

Hence $a_0 = f(0, 0)$, $a_1 = \left(\frac{\partial f}{\partial x}\right)_0$, $a_2 = \left(\frac{\partial f}{\partial y}\right)_0$,

$$a_3 = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_0, \quad a_4 = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_0, \quad a_5 = \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}\right)_0.$$

The idea of a dominant function (§ 28) may also be applied to a function of two or more variables. Thus the function

$$\phi(x, y) = A_{00} + A_{10}x + A_{01}y + \cdots + A_{ik}x^iy^k + \cdots \tag{8}$$

dominates the function

$$f(x, y) = a_{00} + a_{10}x + a_{01}y + \cdots + a_{ik}x^iy^k + \cdots \tag{9}$$

if (8) is a convergent series in which A_{ik} is positive and

$$A_{ik} > |a_{ik}|.$$

If (9) is known to converge for $x = r$, $y = \rho$, where r and ρ are positive, a dominant function is easily found. There is then a number M so that

$$|a_{ik}|r^i\rho^k < M. \tag{10}$$

We place

$$\phi(x, y) = \frac{M}{\left(1 - \frac{x}{r}\right)\left(1 - \frac{y}{\rho}\right)} = \sum M\left(\frac{x}{r}\right)^i\left(\frac{y}{\rho}\right)^k. \tag{11}$$

By (10),

$$|a_{ik}| < \frac{M}{r^i\rho^k},$$

so that (11) dominates (9).

Another dominant function may be found by placing

$$\phi(x, y) = \frac{M}{1 - \left(\frac{x}{r} + \frac{y}{\rho}\right)}$$

$$= M + M\left(\frac{x}{r} + \frac{y}{\rho}\right) + \cdots + M\left(\frac{x}{r} + \frac{y}{\rho}\right)^n + \cdots. \tag{12}$$

The term containing x^iy^k in (12) will be found in the expansion of $M\left(\frac{x}{r} + \frac{y}{\rho}\right)^n$, where $n = i + k$. It is

$$M\,\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{x}{r}\right)^{n-k}\left(\frac{y}{\rho}\right)^k,$$

and therefore, in the notation of (8),

$$A_{ik} = M\,\frac{n(n-1)\cdots(n-k+1)}{k!\,r^i\rho^k}. \tag{13}$$

Now the binomial coefficient which occurs in (13) is always greater than unity, so that

$$A_{ik} > \frac{M}{r^i \rho^k};$$

whence, by (10),

$$A_{ik} > |a_{ik}|$$

and (13) dominates (9).

EXERCISES

Find the first and second partial derivatives of these functions:

1. $\log(x^2 + y^2)$. **2.** $\tan^{-1}\frac{y}{x}$. **3.** $\frac{e^{xy}}{x^2 - y^2}$. **4.** $e^y \sin^{-1}(x - y)$.

Verify $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$ for the following functions:

5. $\frac{x - y}{x + y}$. **6.** $\log\sqrt{x^2 + y^2}$. **7.** $\sin^{-1}\frac{y}{x}$. **8.** $e^x \sin y$.

9. If $f = \frac{1}{x^2 + y^2}\sin(x^2 + y^2)$, prove that $y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} = 0$.

10. If $f = y^2 + 2\,ye^{\frac{1}{x}}$, prove that $x^2\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2\,y^2$.

11. If $f = \sqrt{y^2 - x^2}\,\sin^{-1}\frac{y}{x}$, prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$.

12. If $f = \log(x^2 + y^2) + \tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

13. If $V = e^{a\phi}\cos(a \log r)$, prove that $\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$.

14. If $f = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$, prove that $\frac{\partial^2 f}{\partial x^2} = a^2\frac{\partial^2 f}{\partial y^2}$.

15. If $z = f\left(\frac{y}{x}\right)$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

16. If $z = f(xy)$, prove that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$.

17. If $z = y^2 + 2\,f\left(\frac{1}{x} + \log y\right)$, prove that $\frac{\partial z}{\partial y} = 2\,y - \frac{x^2}{y}\frac{\partial z}{\partial x}$.

18. If $z = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, prove that $x^2\frac{\partial^2 z}{\partial x^2} + 2\,xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$.

19. If $z = \phi(x + iy) + \psi(x - iy)$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

20. If $u = f(x, y)$ and $y = F(x)$, find $\frac{d^2 u}{dx^2}$.

21. If $z = f(x, y)$ and $x = e^u$, $y = e^v$, prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} + x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}.$$

22. If $z = f(x, y)$ and $x = e^u \cos v$, $y = e^u \sin v$, prove that

$$\frac{\partial^2 z}{\partial u\, \partial v} = xy\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) + (x^2 - y^2) \frac{\partial^2 z}{\partial x\, \partial y} - y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y}.$$

23. If $z = e^v \sec u$, $y = e^v \tan u$, and $\phi = \phi(x, y)$, prove that

$$\cos u \left(\frac{\partial^2 \phi}{\partial u\, \partial v} - \frac{\partial \phi}{\partial v}\right) = xy \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + (x^2 + y^2) \frac{\partial^2 \phi}{\partial x\, \partial y}.$$

24. If V is a function of x and y, and $x + y = 2\, e^\theta \cos \phi$, $x - y = 2\, ie^\theta \sin \phi$ $\left(i = \sqrt{-1}\right)$, prove that

$$\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial \phi^2} = 4\, xy \frac{\partial^2 V}{\partial x\, \partial y}.$$

25. Given $x = e^u \cos v$, $y = e^u \sin v$, find $\dfrac{\partial^2 V}{\partial u^2}$ in terms of the derivatives of V with respect to x and y.

26. Given $x = e^u \cos v$, $y = e^u \sin v$, prove that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = e^{-2u}\left(\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right).$$

27. Given $x = u + v$, $y = \dfrac{u - v}{a}$, prove that $a^2 \dfrac{\partial^2 V}{\partial x^2} - \dfrac{\partial^2 V}{\partial y^2} = a^2 \dfrac{\partial^2 V}{\partial u\, \partial v}$.

28. Given $x = r \cosh \theta$, $y = r \sinh \theta$, prove that

$$\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} - \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r} \frac{\partial V}{\partial r}.$$

29. If $x = f(u, v)$ and $y = (u, v)$ are two functions which satisfy the equations

$$\frac{\partial f}{\partial u} = \frac{\partial \phi}{\partial v}, \qquad \frac{\partial f}{\partial v} = -\frac{\partial \phi}{\partial u},$$

and V is any function of x and y, prove that

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$

30. If $u = \dfrac{1}{x^2 + y^2}$, find the contour lines and the direction and magnitude of the gradient.

31. If an electric potential V is given by $V = \log \sqrt{x^2 + y^2}$, find the direction and magnitude of the maximum rate of change of V.

32. Find the direction and magnitude of the gradient of a potential

$$V = \log \frac{\sqrt{(x - a)^2 + y^2}}{\sqrt{(x + a)^2 + y^2}} \text{ at the point } (0, a).$$

33. Find the direction and magnitude of the gradient of

$$u = e^{-y} \sin x + \frac{1}{3} e^{-3y} \sin 3x \text{ at the point } \left(\frac{\pi}{3}, 0\right).$$

34. Show that in polar coördinates the rate of change of a function f along a radius vector is $\frac{\partial f}{\partial r}$, and normal to a vector it is $\frac{1}{r}\frac{\partial f}{\partial \theta}$.

35. Show that in polar coördinates the directional derivative is $\frac{\partial f}{\partial r} \cos \psi + \frac{1}{r}\frac{\partial f}{\partial \theta} \sin \psi$, where ψ is the angle made by the direction considered with the radius from the origin.

36. Show that in polar coördinates the gradient is

$$\sqrt{\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2}.$$

37. Show that the sum of the squares of the directional derivative in any two mutually perpendicular directions is constant and equal to the square of the gradient.

38. If $y = \phi(x)$ is any curve, show that the directional derivative along this curve is

$$\frac{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\phi'(x)}{\sqrt{1 + \phi'^2(x)}}.$$

Show that the following differentials are exact, and find the function of which each is the differential:

39. $(2x - y + 1)dx + (2y - x - 1)dy.$

40. $\frac{1 + y^2}{x^3} dx - \frac{1 + x^2}{x^2} y\, dy.$

41. $\frac{x\, dx}{\sqrt{x^2 + y^2}} + \left(-1 + \frac{y}{\sqrt{x^2 + y^2}}\right) dy.$

42. $\frac{2x - y}{x^2 + y^2} dx + \frac{2y + x}{x^2 + y^2} dy.$

43. $(z^2y^2 - z^2)x\, dx + (x^2z^2 + z^2)y\, dy + (x^2y^2 - x^2 + y^2 - 1)z\, dz.$

44. $(y + z - b - c)dx + (z + x - c - a)dy + (x + y - a - b)dz.$

45. $(y + z)(2x + y + z)dx + (z + x)(2y + z + x)dy$
$+ (x + y)(2z + x + y)dz.$

46. $\left(\frac{1}{y} - \frac{z}{x^2}\right)dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy + \left(\frac{1}{x} - \frac{y}{z^2}\right)dz.$

47. Show that a force directed toward the origin and inversely proportional to the square of the distance from the origin is conservative.

48. Show that a force directed toward a center and equal to any function of the distance from the center is conservative.

CHAPTER IV

IMPLICIT FUNCTIONS

39. One equation, two variables. We are accustomed to say, somewhat roughly, that an equation

$$f(x, y) = 0 \tag{1}$$

defines y as an implicit function of x and is equivalent to the equation

$$y = f(x).$$

We shall proceed to inquire just what this means, to examine the hypothesis underlying the statement, and to put the statement in a more scientific form.

In the most elementary cases equation (1) can be solved to obtain y. For example,

$$x^2 + y^2 - a^2 = 0$$

gives

$$y = \sqrt{a^2 - x^2};$$

but a little reflection shows that as soon as equation (1) becomes complicated such an elementary solution is impossible. We need to show, then, that y in (1) satisfies the definition of a function in the sense that a value of x assumed in (1) determines a value of y. This statement, however, is by no means self-evident. We shall proceed to prove it in the case in which the solution may be expressed as a power series. We shall assume that equation (1) may be satisfied by $x = x_0$, $y = y_0$, and that

$$\left(\frac{\partial f}{\partial y}\right)_{\substack{x = x_0 \\ y = y_0}} \neq 0.$$

Without loss of generality we may take $x_0 = 0$, $y_0 = 0$, as this amounts merely to a change of coördinates. We also assume that $f(x, y)$ may be expanded into a series, so that we may put

$$f(x, y) = b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + \cdots, \tag{2}$$

where the term a_{00} is omitted since, by hypothesis, $f(0, 0) = 0$. Equation (1) becomes

$$b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + \cdots = 0. \tag{3}$$

Now $b_{01} = \left(\frac{\partial f}{\partial y}\right)_0$; and since, by hypothesis, this is not zero, we may divide through by it and, after transposition, have, from (3),

$$y = a_{10}x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots. \tag{4}$$

This means that we have transformed equation (1) into the form

$$y = F(x, y),$$

where $F(x, y)$ is defined by the series in (4).

In (4) let us substitute the series with undetermined coefficients

$$y = c_1x + c_2x^2 + c_3x^3 + \cdots. \tag{5}$$

By equating coefficients of like powers of x in the result the coefficients c_i are easily determined. For the first three we have

$$\begin{aligned} c_1 &= a_{10}, \\ c_2 &= a_{20} + a_{11}c_1 + a_{02}c_1{}^2, \\ c_3 &= a_{30} + a_{21}c_1 + a_{12}c_1{}^2 + a_{03}c_1{}^3 + a_{11}c_2 + 2\,a_{02}c_1c_2. \end{aligned}$$

The series (5) then formally satisfies the equation. It remains to prove that (5) converges for sufficiently small values of x.

The series (4) is, by hypothesis, convergent. Then, with the notation of § 38, the function

$$\phi(x, y) = \frac{M}{\left(1 - \frac{x}{r}\right)\left(1 - \frac{y}{\rho}\right)} - M - \frac{My}{\rho}$$

dominates $F(x, y)$. That is, $\phi(x, y)$ may be expanded into a series with positive coefficients

$$\phi(x, y) = A_{10}x + A_{20}x^2 + A_{11}xy + A_{02}y^2 + \cdots, \tag{6}$$

where $$A_{ik} > |a_{ik}|. \tag{7}$$

If we solve the equation $$y = \phi(x, y) \tag{8}$$

by a series expansion $$y = C_1x + C_2x^2 + \cdots \tag{9}$$

in the same manner in which we have solved (1), the coefficients C_i are obtained from (6) by the same formulas by which the coefficients of c_i in (5) are obtained from (4), and it is evident, therefore, that

$$C_i > |c_i|.$$

Hence the series (5) converges if (9) does. But the series (9) may be obtained by solving (8) as a quadratic equation in y and expanding the result by the binomial theorem. The resulting series is known to converge for sufficiently small values of x and can be no other than (9). Hence the series (5) converges.

The existence of the function y of x having thus been proved, its derivatives may be found by use of the theorems of § 36. For we have by that section

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0; \tag{10}$$

whence

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}. \tag{11}$$

Higher derivatives may be found by differentiating (11), or if we divide (10) by dx and denote $\frac{dy}{dx} = y'$, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\,y' = 0.$$

Then differentiating successively with respect to x and placing $y'' = \frac{d^2y}{dx^2}$, $y''' = \frac{d^3y}{dx^3}$, $\cdots$, etc., we have the equations

$$\frac{\partial^2 f}{\partial x^2} + 2\,\frac{\partial^2 f}{\partial x\,\partial y}\,y' + \frac{\partial^2 f}{\partial y^2}\,y'^2 + \frac{\partial f}{\partial y}\,y'' = 0,$$

$$\frac{\partial^3 f}{\partial x^3} + 3\,\frac{\partial^3 f}{\partial x^2\,\partial y}\,y' + 3\,\frac{\partial^2 f}{\partial x\,\partial y^2}\,y'^2 + \frac{\partial^3 f}{\partial y^3}\,y'^3 + 3\,\frac{\partial^2 f}{\partial y^2}\,y'y''$$
$$+ \frac{\partial^2 f}{\partial x\,\partial y}\,y'' + \frac{\partial f}{\partial y}\,y''' = 0.$$

In this way we may find the derivatives y', y'', y''', etc., provided the partial derivatives of $f(x, y)$ exist, since in all cases the coefficient of each of the derivatives y', y'', y''' in the equation in which it first appears is $\frac{\partial f}{\partial y}$, which has been assumed not to vanish. In this way we may write down series (4) without following the method of the text.

40. One equation, more than two variables. The equation

$$f(x, y, z) = 0 \tag{1}$$

defines any one of the variables, say z, as a function of the other two, x and y; namely,

$$z = \phi(x, y), \tag{2}$$

provided

$$\frac{\partial f}{\partial z} \neq 0.$$

Proof of this statement and its exact formulation are similar to those of § 39 and will not be repeated.

If we apply theorems I and II, § 36, we have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = 0. \tag{3}$$

If in (3) we place $y =$ constant, then $dy = 0$, and we have

$$\left(\frac{dz}{dx}\right)_y = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}}, \tag{4}$$

where the symbol on the left is used as less ambiguous than $\dfrac{\partial z}{\partial x}$, since the variable which is held constant in the differentiation is explicitly given as a subscript.

Similarly, by placing $x =$ constant, $dx = 0$, we have

$$\left(\frac{dy}{dz}\right)_x = -\frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial y}}. \tag{5}$$

We may also in (3) place $z =$ constant, $dz = 0$, and have

$$\left(\frac{dx}{dy}\right)_z = -\frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}}. \tag{6}$$

From (4), (5), and (6) we get the interesting relation

$$\left(\frac{dz}{dx}\right)_y\left(\frac{dx}{dy}\right)_z\left(\frac{dy}{dz}\right)_x = -1,$$

which is sometimes written

$$\frac{\partial z}{\partial x}\,\frac{\partial x}{\partial y}\,\frac{\partial y}{\partial z} = -1. \tag{7}$$

Equation (7) is an example of the fact that the student should not use ∂z, ∂x, etc. as symbols for differentials which may be canceled. To do this in (7) leads to an absurd result. The less ambiguous equation (6) would hardly lend itself to this false cancellation.

The equation
$$f(x, y, z, u, v, \cdots) = 0 \tag{8}$$
defines any one of the variables as a function of the others, provided the derivative of f with respect to the variable chosen does not vanish. The usual hypothesis as to continuity must be made, and the proof of the theorem is analogous to that given in § 39.

From (8) we have
$$f_x\,dx + f_y\,dy + f_z\,dz + f_u\,du + f_v\,dv + \cdots = 0. \tag{9}$$

In (9) we may place all differentials except any two equal to zero. For example, let $dy \neq 0$ and $du \neq 0$, but all others be equal to zero.

Then
$$f_y\,dy + f_u\,du = 0;$$
whence
$$\frac{\partial y}{\partial u} = -\frac{f_u}{f_y},$$
where all variables except the two occurring in the symbol $\frac{\partial y}{\partial u}$ are held constant.

41. Two equations, four variables. Two equations
$$F(x, y, u, v) = 0, \tag{1}$$
$$G(x, y, u, v) = 0, \tag{2}$$
define u and v as functions of x and y, provided
$$F_uG_v - F_vG_u \neq 0. \tag{3}$$

This statement assumes that there is a set of values (x_0, y_0, u_0, v_0) which satisfy equations (1) and (2) in the neighborhood of which F and G are continuous with continuous first derivatives and for which the condition (3) is satisfied. Then u and v are defined as functions of x and y in the neighborhood of (x_0, y_0, u_0, v_0).

To prove this consider equation (1). By virtue of (3), F_u and F_v cannot both be zero. Let us for definiteness assume that $F_v \neq 0$. Then, by § 39, (1) defines v as a function of (x, y, u); namely,
$$v = \phi(x, y, u). \tag{4}$$
From (1),
$$F_x\,dx + F_y\,dy + F_u\,du + F_v\,dv = 0, \tag{5}$$
whence
$$\left(\frac{dv}{du}\right)_{xy} = -\frac{F_u}{F_v};$$
that is,
$$\phi_u = \frac{\partial \phi}{\partial u} = -\frac{F_u}{F_v}. \tag{6}$$

Substitute from (4) in (2). Then G becomes a function of x, y, and u, so that

$$G(x, y, u, v) = H(x, y, u) = 0, \tag{7}$$

which by § 39 defines u as a function of x and y if $H_u \neq 0$. But

$$\begin{aligned} dH = dG &= G_x\,dx + G_y\,dy + G_u\,du + G_v\,dv \\ &= G_x\,dx + G_y\,dy + G_u\,du + G_v(\phi_x\,dx + \phi_y\,dy + \phi_u\,du) \\ &= (G_x + G_v\phi_x)dx + (G_y + G_v\phi_y)dy + (G_u + G_v\phi_u)du\,; \end{aligned}$$

whence, by theorem III, § 36,

$$H_u = G_u + G_v\phi_u = \frac{F_vG_u - F_uG_v}{F_v}, \tag{8}$$

the last reduction being made by (6). Hence the condition that $H_u \neq 0$ is the condition (3).

This condition being fulfilled, (7) defines u as a function of x and y, and then (4) gives v as a function of x and y, and the statement is proved.

The statement having been proved, the partial derivatives of u and v are best found by using the principles of § 36, as follows: From (1) and (2), by I, § 36,

$$F_x\,dx + F_y\,dy + F_u\,du + F_v\,dv = 0, \tag{9}$$

$$G_x\,dx + G_y\,dy + G_u\,du + G_v\,dv = 0. \tag{10}$$

These equations may be solved for du and dv in terms of dx and dy, thus:

$$du = -\frac{(F_xG_v - F_vG_x)dx + (F_yG_v - F_vG_y)dy}{F_uG_v - F_vG_u}, \tag{11}$$

$$dv = -\frac{(F_uG_x - F_xG_u)dx + (F_uG_y - F_yG_u)dy}{F_uG_v - F_vG_u}. \tag{12}$$

It is to be noticed that the denominator in (11) and (12) is just the expression which by hypothesis does not vanish. Should it vanish, equations (9) and (10) cannot be solved for du and dv.

The partial derivatives $\dfrac{\partial u}{\partial x} = \left(\dfrac{du}{dx}\right)_y$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$, $\dfrac{\partial v}{\partial y}$ can then be read off as the coefficients of dx and dy in (11) and (12) by use of III, § 36.

A special case of importance occurs when equations (1) and (2) take the form

$$x = \phi(u, v), \tag{13}$$

$$y = \psi(u, v), \tag{14}$$

where the connection with (1) and (2) is shown by placing $F=x-\phi(u, v)$, $G=y-\psi(u, v)$. Then the condition that (1) and (2) may be solved for u and v is

$$\phi_u\psi_v-\phi_v\psi_u \neq 0. \tag{15}$$

From (13) and (14) we form

$$dx=\phi_u\,du+\phi_v\,dv,$$
$$dy=\psi_u\,du+\psi_v\,dv;$$

whence
$$du=\frac{\psi_v\,dx-\phi_v\,dy}{\phi_u\psi_v-\phi_v\psi_u},$$
$$dv=\frac{-\psi_u\,dx+\phi_u\,dy}{\phi_u\psi_v-\phi_v\psi_u}; \tag{16}$$

whence we find
$$\left(\frac{du}{dx}\right)_y=\frac{\psi_v}{\phi_u\psi_v-\phi_v\psi_u}, \tag{17}$$

and similar expressions for $\left(\frac{du}{dy}\right)_x$, $\left(\frac{dv}{dx}\right)_y$, $\left(\frac{dv}{dy}\right)_x$.

Now $\left(\frac{du}{dx}\right)_y$ may be written, when no ambiguity is caused, as $\frac{\partial u}{\partial x}$ and $\phi_u=\frac{\partial x}{\partial u}$, $\phi_v=\frac{\partial x}{\partial v}$, $\psi_u=\frac{\partial y}{\partial u}$, $\psi_v=\frac{\partial y}{\partial v}$.

The relation (17) shows emphatically that $\frac{\partial u}{\partial x}$ is not the reciprocal of $\frac{\partial x}{\partial u}$.

42. Three equations, six variables. Consider the equations

$$F(x, y, z, u, v, w)=0, \tag{1}$$
$$G(x, y, z, u, v, w)=0, \tag{2}$$
$$H(x, y, z, u, v, w)=0. \tag{3}$$

By § 40, equation (1) defines w as a function of x, y, z, u, v, provided $F_w \neq 0$, and we have

$$F_x\,dx+F_y\,dy+F_z\,dz+F_u\,du+F_v\,dv+F_w\,dw=0. \tag{4}$$

If the value of w is substituted in (2) and (3), we have two equations which may be treated as in § 41 and solved for u and v in terms of x, y, and z, provided the condition of § 41 is satisfied. This condition is

$$\left(G_u+G_w\frac{dw}{du}\right)\left(H_v+H_w\frac{dw}{dv}\right)-\left(G_v+G_w\frac{dw}{dv}\right)\left(H_u+H_w\frac{dw}{du}\right)\neq 0,$$

where $\frac{dw}{du}$ and $\frac{dw}{dv}$ are to be obtained from (4). This gives

$$F_u \frac{G_v H_w - G_w H_v}{F_w} + F_v \frac{G_w H_u - G_u H_w}{F_w} + F_w \frac{G_u H_v - G_v H_u}{F_w} \neq 0, \tag{5}$$

or, in determinant form,

$$\frac{1}{F_w} \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix} \neq 0. \tag{6}$$

We have obtained this result on the hypothesis that $F_w \neq 0$. The same result is obtained, as the student may verify, if we assume either that $F_u \neq 0$ or that $F_v \neq 0$. If all three of the derivatives F_w, F_u, F_v are zero, condition (5) is certainly not satisfied. Hence we have the result that the original equations determine u, v, and w as functions of x, y, and z, provided (5) is satisfied.

The partial derivatives of u, v, and w with respect to x, y, and z are most readily found by considering, together with (4), the equations

$$G_x\,dx + G_y\,dy + G_z\,dz + G_u\,du + G_v\,dv + G_w\,dw = 0, \tag{7}$$

$$H_x\,dx + H_y\,dy + H_z\,dz + H_u\,du + H_v\,dv + H_w\,dw = 0, \tag{8}$$

solving (4), (7), and (8) for du, dv, dw, and applying III, § 36.

A special case of importance occurs when the original equations are in the form

$$\begin{aligned} x &= \phi(u, v, w), \\ y &= \psi(u, v, w), \\ z &= \chi(u, v, w), \end{aligned} \tag{9}$$

and our work shows that these may be solved for u, v, w if

$$\begin{vmatrix} \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{vmatrix} \neq 0.$$

43. The general case. If we have n variables connected by p equations ($n \geqq p$), there are in general $n - p$ independent variables which may be taken at pleasure. The remaining p variables are functions of the others. The derivatives may be found by applying to each equation theorems I and II of § 36, solving for the differentials of the functions, and applying theorem III of § 36.

As noted in the words "in general" there may be exceptions to the existence of the functions. These exceptions will be characterized by the vanishing of certain combinations of derivatives, while at the same time the solution for the differentials becomes impossible. If $n < p$ the equations will be in general contradictory.

We shall not take the space to prove these statements. The general method of proof is sufficiently evident from the simpler cases already handled.

44. Jacobians. Return to the equations

$$x = \phi(u, v), \tag{1}$$

$$y = \psi(u, v), \tag{2}$$

of § 41.

The expression

$$\phi_u\psi_v - \phi_v\psi_u, \tag{3}$$

which figures there, is called the *Jacobian*, or *functional determinant*, of x and y with respect to u and v, and is variously expressed by the symbols

$$J\frac{(x, y)}{(u, v)} = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}. \tag{4}$$

We have proved in § 41 the theorem

I. The necessary and sufficient condition that equations (1) *and* (2) *should be solvable for u and v is that the Jacobian* (4) *should not vanish.*

We shall now show that if the Jacobian vanishes, there is a functional relation between x and y. If all the derivatives $\phi_u, \phi_v, \psi_u, \psi_v$ are zero, equations (1) and (2) reduce to the trivial case $x =$ constant, $y =$ constant. We may therefore assume that at least one of them (for definiteness say ϕ_v) does not vanish.

Then (1) defines v as a function of x and u, and

$$dx = \phi_u\, du + \phi_v\, dv;$$

whence

$$dv = \frac{1}{\phi_v}\, dx - \frac{\phi_u}{\phi_v}\, du. \tag{5}$$

Then from (2), with v expressed as a function of x and u,

$$\begin{aligned} dy &= \psi_u\, du + \psi_v\left(\frac{1}{\phi_v}\, dx - \frac{\phi_u}{\phi_v}\, du\right) \\ &= \frac{\psi_u\phi_v - \psi_v\phi_u}{\phi_v}\, du + \frac{\psi_v}{\phi_v}\, dx. \end{aligned} \tag{6}$$

By hypothesis the coefficient of du in (6) vanishes. Hence if y is expressed as a function of u and x, $\frac{\partial y}{\partial u} = 0$, and y is independent of u and a function of x only.

Conversely, let us assume that in (1) there is a functional relation between x and y; namely,

$$F(x, y) = 0. \tag{7}$$

Then
$$F_x\,dx + F_y\,dy = 0; \tag{8}$$

whence
$$(F_x\phi_u + F_y\psi_u)du + (F_x\phi_v + F_y\psi_v)dv = 0. \tag{9}$$

The last equation is true for all values of the independent variables u and v. Hence

$$F_x\phi_u + F_y\psi_u = 0, \qquad F_x\phi_v + F_y\psi_v = 0; \tag{10}$$

whence
$$\phi_u\psi_v - \phi_v\psi_u = 0. \tag{11}$$

Therefore we have proved the following theorem:

II. The necessary and sufficient condition that a functional relation should exist between x and y in (1) *is that the Jacobian* $\frac{\partial(x, y)}{\partial(u, v)}$ *should vanish.*

As a simple illustration of these theorems consider the equations

$$x = au + bv + c,$$
$$y = fu + gv + h.$$

The functional determinant then becomes $ag - bf$. Now if $ag - bf \neq 0$, the equations can be solved for u and v. But if $ag - bf = 0$, the equations cannot be solved. In fact, in this case

$$gx - by = cg - bh,$$

a functional relation between x and y.

As a second example, consider

$$x = u + v + 1,$$
$$y = u^2 + 2\,uv + v^2 + 2.$$

Here
$$J\left(\frac{x,\ y}{u,\ v}\right) = \begin{vmatrix} 1 & 1 \\ 2\,u + 2\,v & 2\,u + 2\,v \end{vmatrix} = 0,$$

and the equations cannot be solved for u and v. Obviously

$$y = (x - 1)^2 + 2,$$

so that there is a functional relation between x and y.

The following property of the Jacobian is important; namely,

$$J\left(\frac{x,\ y}{u,\ v}\right)\cdot J\left(\frac{u,\ v}{x,\ y}\right)=1. \tag{12}$$

Writing the left-hand member in determinant form and combining by the law of multiplication of determinants,* we have

$$\begin{vmatrix} \phi_u & \phi_v \\ \psi_u & \psi_v \end{vmatrix} \cdot \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \phi_u \dfrac{\partial u}{\partial x} + \phi_v \dfrac{\partial v}{\partial x} & \phi_u \dfrac{\partial u}{\partial y} + \phi_v \dfrac{\partial v}{\partial y} \\ \psi_u \dfrac{\partial u}{\partial x} + \psi_v \dfrac{\partial v}{\partial x} & \psi_u \dfrac{\partial u}{\partial y} + \psi_v \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \psi}{\partial x} & \dfrac{\partial \psi}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

In a similar manner it may be proved that

$$J\left(\frac{u,\ v}{x,\ y}\right)\cdot J\left(\frac{x,\ y}{\xi,\ \eta}\right)=J\left(\frac{u,\ v}{\xi,\ \eta}\right). \tag{13}$$

Again, if we have, as in (9), § 42,

$$x=\phi(u,\ v,\ w), \tag{14}$$

$$y=\psi(u,\ v,\ w), \tag{15}$$

$$z=\chi(u,\ v,\ w), \tag{16}$$

the determinant

$$\begin{vmatrix} \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{vmatrix} \tag{17}$$

is called the Jacobian of x, y, and z with respect to u, v, and w and is expressed by the symbol

$$J\left(\frac{x,\ y,\ z}{u,\ v,\ w}\right)=\frac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}. \tag{18}$$

The results of § 42 may then be expressed by the theorem

III. The necessary and sufficient condition that (14), (15), (16) *may be solved for u, v, w is that the Jacobian of x, y, and z with respect to u, v, and w shall not vanish.*

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* Students to whom determinants are unfamiliar may verify this by actual multiplication, using simply the definition that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a symbol for the expression $a_1b_2 - a_2b_1$.

Suppose now the Jacobian does vanish. If all the derivatives ϕ_u, ϕ_v, ϕ_w vanish, equation (14) reduces to the trivial case $x = \text{constant}$. We shall therefore assume that at least one of these derivatives does not vanish and shall take $\phi_w \neq 0$ for definiteness. Then (14) may be solved for w, and the result be substituted in (15) and (16). By the same process which obtained (5), § 42, we see that the Jacobian of y and z with respect to u and v from these equations is

$$\frac{1}{\phi_w} J\left(\frac{x,\ y,\ z}{u,\ v,\ w}\right),$$

which vanishes by hypothesis. Hence, by the earlier part of this section, there is a functional relation between y and z. This relation obviously may, and usually does, contain x, which in this work has been considered merely as a parameter.

Conversely, let a functional relation

$$F(x, y, z) = 0 \tag{19}$$

exist between x, y, and z of (14), (15), (16). This relation exists for all values of the independent variables u, v, w, and therefore the partial derivatives of F with respect to u, v, and w vanish. Hence

$$F_x \frac{\partial x}{\partial u} + F_y \frac{\partial y}{\partial u} + F_z \frac{\partial z}{\partial u} = 0,$$
$$F_x \frac{\partial x}{\partial v} + F_y \frac{\partial y}{\partial v} + F_z \frac{\partial z}{\partial v} = 0,$$
$$F_x \frac{\partial x}{\partial w} + F_y \frac{\partial y}{\partial w} + F_z \frac{\partial z}{\partial w} = 0.$$

By a well-known theorem of algebra this equation can exist when and only when the determinant of the coefficients of F_x, F_y, and F_z vanishes. But this determinant is the Jacobian. Hence

IV. The necessary and sufficient condition that a functional relation should exist between x, y, and z is that the Jacobian of x, y, and z with respect to u, v, and w should vanish.

EXERCISES

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ from each of the following equations:

1. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

2. $x^n + y^n = a^n$.

3. $e^x + e^y = e^{x+y}$.

4. $\log(x^2 + y^2) - \tan^{-1}\dfrac{y}{x} = 0$.

5. $\cos(x + y) + \cos(x - y) = 1$.

6. $e^{x+y} = y^x$.

7. If $f(x, y) = 0$, prove that

$$\frac{d^2y}{dx^2} = -\frac{\dfrac{\partial^2 f}{\partial x^2}\left(\dfrac{\partial f}{\partial y}\right)^2 - 2\dfrac{\partial^2 f}{\partial x\,\partial y}\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y} + \dfrac{\partial^2 f}{\partial y^2}\left(\dfrac{\partial f}{\partial x}\right)^2}{\left(\dfrac{\partial f}{\partial y}\right)^3}.$$

Find $\dfrac{\partial z}{\partial x}$ from each of the following equations:

8. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

9. $x + y + z = kxyz.$

10. $x^3 + y^3 + z^3 - u^3 = 0.$

11. $\sin^{-1}\dfrac{y}{z} + \log(x^2 + y^2 + z^2) = 0.$

12. If $f(x, y, z, u) = 0$, prove that

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial u} = 1,$$

where each partial derivative is found on the hypothesis that all variables except the two involved are constant.

13. If $f(x, y, z, u) = 0$, prove that

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial u} = 1,$$

where each partial derivative is found as in Ex. 12.

14. If $f(x, y, z, u) = 0$, prove that

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1,$$

where each partial derivative is found on the hypothesis that all variables except the two involved are constant.

15. Given that an equation $f(v, p, t) = 0$ connects the volume, pressure, and temperature of a gas, that $\alpha_p = \dfrac{1}{v}\left(\dfrac{dv}{dt}\right)_p$ is the coefficient of expansion, and that $E_t = -v\left(\dfrac{dp}{dv}\right)_t$ is the modulus of elasticity, prove that $\alpha_p E_t$ is equal to the rate of increase of pressure with respect to the temperature if the volume is constant.

16. Given $x = r\cosh\theta$, $y = r\sinh\theta$, prove that

$$\left(\frac{dx}{dr}\right)_\theta = \left(\frac{dr}{dx}\right)_y,$$

$$\left(\frac{dy}{dr}\right)_\theta = -\left(\frac{dr}{dy}\right)_x.$$

17. Given $u = \log\sqrt{x^2 + y^2}$, $v = \tan^{-1}\dfrac{y}{x}$, prove that

$$\left(\frac{du}{dx}\right)_y \left(\frac{dx}{du}\right)_v + \left(\frac{du}{dy}\right)_x \left(\frac{dy}{du}\right)_v = 1.$$

18. If $u = \phi(x, y)$, $v = \psi(x, y)$, prove that

$$\left(\frac{du}{dx}\right)_y \left(\frac{dx}{du}\right)_v + \left(\frac{dv}{dx}\right)_y \left(\frac{dx}{dv}\right)_u = 1.$$

19. If $u = \phi(x, y)$, $v = \psi(x, y)$, prove that

$$\left(\frac{du}{dx}\right)_y \left(\frac{dy}{du}\right)_v + \left(\frac{dv}{dx}\right)_y \left(\frac{dy}{dv}\right)_u = 0.$$

20. If $f(u, v) = 0$, $u = lx + my + nz$, $v = x^2 + y^2 + z^2$, prove that

$$(ly - mx) + (ny - mz)\frac{\partial z}{\partial x} + (lz - nx)\frac{\partial z}{\partial y} = 0.$$

21. If $f(u, v) = 0$, $u = \dfrac{y}{x}$, $v = \dfrac{z^2}{x}$, prove that

$$2\,x\frac{\partial z}{\partial x} + 2\,y\frac{\partial z}{\partial y} = z.$$

22. If $f(x^2 - y^2, y^2 - z^2) = 0$, prove that

$$yz\frac{\partial z}{\partial x} + zx\frac{\partial z}{\partial y} = xy.$$

23. If $f\left(\dfrac{y}{x}, \dfrac{z}{x}\right) = 0$, prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

24. Given $u^3 + v^3 + x^3 - 3\,y = 0$, $u^2 + v^2 + y^2 + 2\,x = 0$, find $\left(\dfrac{du}{dx}\right)_y$.

25. Given $x - y + u - v = a$, $x - y + u - v = b$, find $\left(\dfrac{dv}{dy}\right)_x$.

26. Show that $F(x, y, z)$, $G(x, y, z) = 0$ define x and y as functions of z, and prove that

$$dx : dy : dz = F_yG_z - F_zG_y : F_zG_x - F_xG_z : F_xG_y - F_yG_x.$$

27. If $x = \phi(u, v)$, $y = \psi(u, v)$, $z = \chi(u, v)$, show that in general $z = f(x, y)$ and prove that

$$\left(\frac{dz}{dx}\right)_y = J\left(\frac{y,\ z}{u,\ v}\right) \div J\left(\frac{x,\ y}{u,\ v}\right).$$

28. Given $x = r\cosh\theta$, $y = r\sinh\theta$, find $\left(\dfrac{\partial f}{\partial x}\right)^2 - \left(\dfrac{\partial f}{\partial y}\right)^2$ in terms of r, θ, $\dfrac{\partial f}{\partial r}$, $\dfrac{\partial f}{\partial \theta}$.

29. Given $z = f\left(\frac{x-y}{y}\right)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$

30. If $z = f\left(\frac{x^2+y^2}{y^2}\right)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$

31. Given $x + y + u + v = a$, $x^2 + y^2 + u^2 + v^2 = b^2$, find $\left(\frac{dx}{du}\right)_y$.

32. Given $x^2 + y^2 + u^2 - v^2 = 0$, $xy + uv = 0$, find $\left(\frac{du}{dx}\right)_y$.

33. Given $f(x + y - z, x^2 - y^2 + z^2) = 0$, find $\left(\frac{dz}{dx}\right)_y$.

CHAPTER V

APPLICATIONS TO GEOMETRY

45. Element of arc. From a fixed origin O three axes OX, OY, OZ are drawn mutually at right angles determining three mutually perpendicular planes. The coördinates (x, y, z) of a point P are the three perpendicular distances from these planes to P, a distance being positive if measured in the direction of the corresponding axis and negative if measured in the opposite direction.

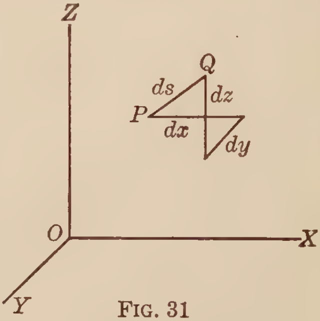

FIG. 31

From P (Fig. 31) let lines of infinitesimal lengths dx, dy, dz be drawn parallel to the axes, thus determining a point Q with coördinates

$$(x + dx,\ y + dy,\ z + dz).$$

Let ds be the length of the infinitesimal line PQ. Then we have

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{1}$$

as is readily seen. This defines the *element of arc*.

The direction of PQ is determined by the angles it makes with dx, dy, dz respectively. Let these angles be α, β, γ respectively.

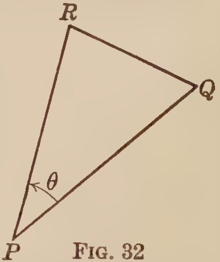

FIG. 32

It is apparent from the figure that

$$\cos\alpha = \frac{dx}{ds}, \quad \cos\beta = \frac{dy}{ds}, \quad \cos\gamma = \frac{dz}{ds}. \tag{2}$$

These are the *direction cosines* of PQ. From (2) and (1) it follows that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{3}$$

It is obvious that a direction is completely fixed by the ratios $dx : dy : dz$.

Let two directions PQ and PR (Fig. 32) be determined by $dx : dy : dz$ and $\delta x : \delta y : \delta z$ respectively, and let θ be the angle between them. Then

$$\cos\theta = \frac{dx}{ds}\frac{\delta x}{\delta s} + \frac{dy}{ds}\frac{\delta y}{\delta s} + \frac{dz}{ds}\frac{\delta z}{\delta s}. \tag{4}$$

This follows from the formula of trigonometry,

$$\overline{RQ}^2 = \overline{PQ}^2 + \overline{PR}^2 - 2\, PQ \cdot PR \cos\theta,$$

where

$$\overline{PQ}^2 = ds^2 = dx^2 + dy^2 + dz^2, \quad \overline{PR}^2 = \delta s^2 = \delta x^2 + \delta y^2 + \delta z^2,$$
$$\overline{RQ}^2 = (dx - \delta x)^2 + (dy - \delta y)^2 + (dz - \delta z)^2.$$

From (4) it follows that the necessary and sufficient condition that two directions $dx : dy : dz$ and $\delta x : \delta y : \delta z$ should be orthogonal is

$$dx\, \delta x + dy\, \delta y + dz\, \delta z = 0. \tag{5}$$

Let the point P so vary as to describe a curve defined by the equations

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t), \tag{6}$$

where t is an independent variable and the functions are continuous and differentiable. Let $P(x, y, z)$ (Fig. 33) be a point corresponding to a certain value of t, and Q the point obtained by giving to t an increment $\Delta t = dt$. The coördinates of Q are then $(x + \Delta x, y + \Delta y, z + \Delta z)$.

Draw the chord PQ. The direction of this chord is determined by the ratios $\Delta x : \Delta y : \Delta z$. Let $\Delta t \rightarrow 0$; then the ratios $\Delta x : \Delta y : \Delta z$ approach the limiting ratios $dx : dy : dz$. The straight line PT with this direction is the tangent to the curve at P by definition, since it is the limit line approached by a secant through two points on the curve as the two points approach coincidence.

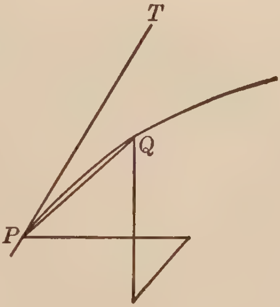

FIG. 33

The point $(x + dx, y + dy, z + dz)$ is then a point on the tangent line; but, except for infinitesimals of higher order than dx, dy, and dz, its coördinates agree with those of Q. The expression (1) then rigorously defines the length of an infinitesimal tangent. However, the length of the curve is to be taken as

$$s = \int \sqrt{dx^2 + dy^2 + dz^2}. \tag{7}$$

This agrees with the definition of the length of a curve as the limit of the sum of the lengths of chords, since $\sqrt{dx^2 + dy^2 + dz^2}$ differs from $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$, the length of a chord, by infinitesimals of higher order, and therefore, by § 13, the integral (7) is the length of the curve.

We may therefore apply our formulas (1), (2), (3), (4) to curves and, in Fig. 31, regard PQ either as an infinitesimal arc or as a tangent line or as a chord.

46. Straight line. Let the point P in Fig. 31, § 45, traverse a straight line. It is evident that the angles α, β, γ are constant. We have then as differential equations for a straight line

$$\frac{dx}{ds} = l, \qquad \frac{dy}{ds} = m, \qquad \frac{dz}{ds} = n, \tag{1}$$

where l, m, and n are three constants (the direction cosines of the line) satisfying the condition

$$l^2 + m^2 + n^2 = 1. \tag{2}$$

Integrating (1) we have, as equations of a straight line,

$$x = ls + x_0, \qquad y = ms + y_0, \qquad z = ns + z_0. \tag{3}$$

Here x_0, y_0, z_0 are constants of integration and are obviously the coördinates of the point from which s is measured, which may be any point of the line.

From (3) we obtain

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} \tag{4}$$

as equations of a line not containing s, but where l, m, n are bound by conditions (2). We may, however, replace l, m, n by any three numbers A, B, C such that

$$l : m : n = A : B : C. \tag{5}$$

Then (4) becomes

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}, \tag{6}$$

and A, B, C are subject to no conditions.

Conversely, if (6) is given we may obtain $l : m : n$ from (5). Then, by (2),

$$l = \frac{A}{\sqrt{A^2 + B^2 + C^2}}, \quad m = \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \quad n = \frac{C}{\sqrt{A^2 + B^2 + C^2}}. \tag{7}$$

From (6) and (5) also

$$dx : dy : dz = A : B : C = l : m : n. \tag{8}$$

That is, *the direction of a straight line may be fixed by the ratios of any three numbers* $A : B : C$. *The direction cosines of the line are then found by* (7).

From (8) of this section and (5), § 45, it follows that the necessary and sufficient condition that two lines with directions $A_1 : B_1 : C_1$ and $A_2 : B_2 : C_2$ should be perpendicular is

$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \tag{9}$$

47. Surfaces. Consider the equation

$$F(x, y, z) = 0. \tag{1}$$

By § 40 this defines one of the variables as a function of the other two. The geometric locus of (1) is therefore a two-dimensional extent which by definition is a surface, whether the equation is satisfied by real or by imaginary values of the variables. For example, the equation

$$x^2 + y^2 + z^2 + 1 = 0$$

is said to define a surface, though no point is real.

If one of the variables is absent from (1) so that, for example, it becomes

$$F(x, y) = 0, \tag{2}$$

it still defines a two-dimensional extent, since y is a function of x, whereas z may vary at pleasure. In fact, if z is placed equal to zero, (2) defines a curve in the XOY plane; but as z varies, that curve is moved parallel to OZ, and the complete locus of (2) is a cylinder.

Finally, if only one of the variables is present in (1) so that, for example, it becomes

$$F(x) = 0, \tag{3}$$

it still defines a two-dimensional extent, since y and z may vary at pleasure. In fact, (3) defines certain values of x, and the complete locus of (3) consists of all those points for which x has one or another of these values. Evidently these points lie on planes parallel to XOY.

Hence equation (1) always represents a surface. On the surface (1) draw any curve with equations

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t). \tag{4}$$

These values of x, y, z placed in (1) must reduce it to identity in t. Hence $\dfrac{dF}{dt} = 0$, and therefore

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0, \tag{5}$$

where $dx : dy : dz$ are subject only to the condition of being the direction of some curve on the surface. Otherwise expressed, $dx : dy : dz$ is any direction on the surface.

Consider any point P on the surface. At that point $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$ have definite values and may be used to fix the direction of a line PN (Fig. 34). That is, we may construct from P directions $\delta x : \delta y : \delta z = \frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}$. Then, from (5), § 45, the line PN is orthogonal to all directions on the surface. This line is called the normal to the surface, and we repeat explicitly that *the direction of the normal to a surface is* $\frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z}$.

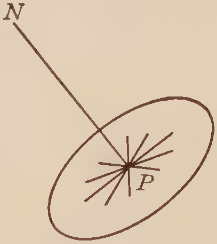

FIG. 34

48. Planes. It is obvious that the necessary and sufficient condition that a surface should be a plane is that the normals at all points should be parallel. Hence a surface is a plane when, and only when,

$$\frac{\partial F}{\partial x} = A, \quad \frac{\partial F}{\partial y} = B, \quad \frac{\partial F}{\partial z} = C, \tag{1}$$

where A, B, C are any constants.

If we integrate the first equation in (1), we have

$$F = Ax + \phi(y, z),$$

where ϕ is an unknown function of y and z. Substituting in the second equation of (1), we have

$$\frac{\partial \phi}{\partial y} = B;$$

whence $$\phi = By + \psi(z).$$

Substituting in the last equation of (1), we have

$$\frac{\partial \psi}{\partial z} = C;$$

whence $$\psi = Cz + D,$$

where D is a constant.

Hence the solution of equations (1) is

$$F = Ax + By + Cz + D,$$

and the surfaces for which (1) holds are

$$Ax + By + Cz + D = 0. \tag{2}$$

Conversely, if (2) is given with any value of the coefficients, (1) follows. Hence

The necessary and sufficient condition that a surface should be a plane is that its equation should be of the form

$$Ax + By + Cz + D = 0.$$

Then $A:B:C$ fix the direction of any normal to the plane.

If (x_1, y_1, z_1) are the coördinates of a point on a plane, they will satisfy (2). Subtracting the resulting equation from (2), we have

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \tag{3}$$

as the equation of a plane through a fixed point perpendicular to a fixed direction or to a fixed straight line.

By (6), § 46, the equations of the normal to (2) through the point (x_1, y_1, z_1) are

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}. \tag{4}$$

Applying this to the surface (1), § 47, we define the tangent plane to the surface at a point as the plane perpendicular to the normal to the surface at the point. Its equation is, therefore, by (3),

$$\left(\frac{\partial F}{\partial x}\right)_1 (x - x_1) + \left(\frac{\partial F}{\partial y}\right)_1 (y - y_1) + \left(\frac{\partial F}{\partial z}\right)_1 (z - z_1) = 0; \tag{5}$$

and the equations of the normal are

$$\frac{x - x_1}{\left(\frac{\partial F}{\partial x}\right)_1} = \frac{y - y_1}{\left(\frac{\partial F}{\partial y}\right)_1} = \frac{z - z_1}{\left(\frac{\partial F}{\partial z}\right)_1}, \tag{6}$$

where $\left(\frac{\partial F}{\partial x}\right)_1$ etc. represent the value of $\frac{\partial F}{\partial x}$ etc. when $x = x_1$, $y = y_1$, $z = z_1$.

A plane is determined by three points not in the same straight line. Let (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) be three such points.

Since the plane passes through (x_1, y_1, z_1), its equation must be of the form

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0. \tag{7}$$

But the points (x_2, y_2, z_2) and (x_3, y_3, z_3) must also lie on the plane. Hence A, B, C in (7) must satisfy the equations

$$\begin{aligned} A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) &= 0, \\ A(x_3 - x_1) + B(y_3 - y_1) + C(z_3 - z_1) &= 0. \end{aligned} \tag{8}$$

If the ratios $A:B:C$ are found from (8) and substituted in (7), the result is the equation of a plane through three points. In determinant notation the result is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0. \tag{9}$$

49. Behavior of a surface near a point. If the equation (1) of § 47 contains z, it may be put in the form

$$z = f(x, y), \tag{1}$$

and obviously any surface may be so expressed by proper choice of the z-axis. For convenience we shall use the common notation by which

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x\, \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2}. \tag{2}$$

The direction of the normal to (1) is, by § 47, $p:q:-1$. The equation of the tangent plane at (a, b, c) is

$$z - c = p(x-a) + q(y-b), \tag{3}$$

and the equations of the normal line are

$$\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{-1}, \tag{4}$$

where in (3) and (4) p and q are to be computed for the point (a, b, c).

We expand the function $f(x, y)$ in the neighborhood of $x = a$, $y = b$ by Taylor's theorem and have, since $c = f(a, b)$,

$$\begin{aligned} z = c + p(x-a) + q(y-b) + \tfrac{1}{2}[r(x-a)^2 \\ + 2\,s(x-a)(y-b) + t(y-b)^2] + R. \end{aligned} \tag{5}$$

The right-hand member gives the distance from the plane XOY to the surface; call it z_2. On the other hand, the value of z in (3) gives the distance from XOY to the tangent plane; call it z_1. Then $z_2 - z_1$ is the distance between the surface and its tangent plane, and

$$z_2 - z_1 = \tfrac{1}{2}[r(x-a)^2 + 2\,s(x-a)(y-b) + t(y-b)^2] + R. \tag{6}$$

We shall study the sign of the expression (6); for if it is always of the same sign, the surface is on the same side of the tangent plane, and if its sign changes, the surface is sometimes on one side of the plane and sometimes on the other side.

On the XOY plane draw a circle (Fig. 35) with $M(a, b)$ as a center. Take P any point inside the circle. Let ρ be the distance MP and let θ be the angle made by MP with MX' parallel to OX. Then

$$x - a = \rho \cos \theta,$$
$$y - b = \rho \sin \theta.$$

In fact, (ρ, θ) are polar coördinates with the origin M. Substituting in (6), we have

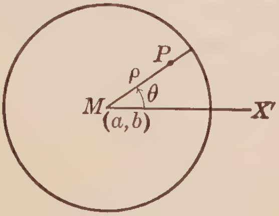

FIG. 35

$$z_2 - z_1 = \tfrac{1}{2}\,\rho^2[r \cos^2\theta + 2\,s \cos\theta \sin\theta + t \sin^2\theta] + \frac{\rho^3}{6} R', \qquad (7)$$

since R is cubic in $x - a$ and $y - b$.

The coefficient of ρ^2 is zero when θ satisfies the equation

$$r \cos^2\theta + 2\,s \cos\theta \sin\theta + t \sin^2\theta = 0; \qquad (8)$$

that is, when
$$\tan\theta = \frac{-s \pm \sqrt{s^2 - rt}}{t}. \qquad (9)$$

CASE I. $rt - s^2 > 0$. The values of $\tan\theta$ in (9) are imaginary. Therefore (8) has no real roots, and the coefficient of ρ^2 in (7) has always the same sign; namely, the sign of r or of t (r and t have the same sign, since $rt > s^2$). It follows that within the circle we have drawn there is some finite quantity m such that

$$|r \cos^2\theta + 2\,s \sin\theta \cos\theta + t \sin^2\theta| > m.$$

Within the same circle, R', in formula (7), must be finite from the nature of the remainder R. Hence there is a number M such that

$$|R'| < M.$$

If we now take ρ so that $\rho < \dfrac{3\,m}{M}$, we have

$$\rho < \frac{3|r \cos^2\theta + 2\,s \sin\theta \cos\theta + t \sin^2\theta|}{|R'|};$$

that is, $\left|R'\right|\dfrac{\rho^3}{6} < \dfrac{\rho^2}{2}\left|r \cos^2\theta + 2\,s \sin\theta \cos\theta + t \sin^2\theta\right|,$

and therefore the sign of $z_2 - z_1$ in (7) is the same as the sign of its first term. Hence

If $rt - s^2 > 0$, the surface lies entirely on one side of its tangent plane and lies above it if r and t are positive and below it if r and t are negative.

The word "above" in the foregoing theorem means in the positive direction of OZ.

Suppose, now, that the origin is transformed to the point (a, b, c) and the tangent plane is taken at the plane XOY. The equation of the surface is then

$$z = \tfrac{1}{2}(rx^2 + 2\,sxy + ty^2) + R,$$

where R involves terms of the third and higher degrees in x and y. Suppose the surface is cut by a plane $z = \epsilon$. The section is the curve

$$rx^2 + 2\,sxy + ty^2 = 2\,\epsilon, \tag{10}$$

neglecting the terms in R. Since $rt - s^2 > 0$, the curve (10) is an ellipse. Hence

When $rt - s^2 > 0$, the section made by a plane parallel to the tangent plane is approximately an ellipse.

For this reason the point is called an elliptic point (Fig. 36). The ellipse

$$rx^2 + 2\,sxy + ty^2 = 1,$$

similar to (10), is called the *indicatrix*.

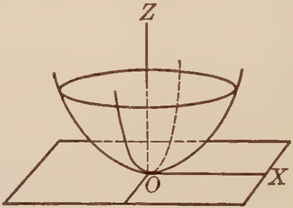

FIG. 36

CASE II. $rt - s^2 < 0$. The values of θ in (9) are real. Call them α_1 and α_2. Then the coefficient of ρ^2 in (7) is

$$\tfrac{1}{2}\,r(\cos\theta - \cot\alpha_1\sin\theta)(\cos\theta - \cot\alpha_2\sin\theta). \tag{11}$$

The real lines $\theta = \alpha_1$ and $\theta = \alpha_2$ (Fig. 37) divide the plane into four portions within which the expression (11) is alternately plus and minus.

Consider a point P with coördinates (ρ, θ_1). Then, from (7),

$$z_2 - z_1 = \tfrac{1}{2}\,\rho^2 A + \frac{\rho^3}{6}\,R', \tag{12}$$

where

$$A = r\cos^2\theta_1 + 2\,s\cos\theta_1\sin\theta_1 + t\sin^2\theta_1.$$

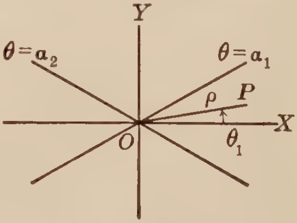

FIG. 37

As $\rho \to 0$ the sign of the first term of (12) determines the sign of the $z_2 - z_1$. But the sign of A depends upon the section of the plane in which P lies. Hence

If $rt - s^2 < 0$, the surface lies partly on one side of the tangent plane and partly on the other side (Fig. 38).

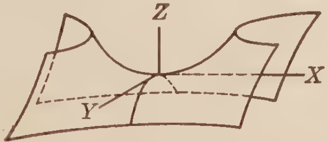

FIG. 38

Let the origin be transformed to the point (a, b, c), and the tangent plane be taken as the plane of XOY. Then the equation of the surface is

$$z = \tfrac{1}{2}(rx^2 + 2\,sxy + ty^2) + R.$$

The section by $z = \epsilon$ is approximately

$$rx^2 + 2\,sxy + ty^2 = 2\,\epsilon,$$

which is a hyperbola, since $rt - s^2 < 0$. Hence

If $rt - s^2 < 0$, the section made by a plane parallel to the tangent plane is approximately a hyperbola.

The point is therefore a *hyperbolic* point. The curve

$$rx^2 + 2\,sxy + ty^2 = 1$$

is the indicatrix.

CASE III. $rt - s^2 = 0$. The coefficient of ρ^2 in (7) is a perfect square, and the roots of (8) are equal. Neither of the arguments made in I and II is valid. The case is ambiguous, and the surface may be either on one side of the tangent plane or on the other. Examples will show this.

Take first $$z = x^2.$$

Here the origin is the point (a, b, c). The tangent plane is $z = 0$, and $r = 2$, $s = 0$, $t = 0$. Since z is always positive, the surface lies on one side of the tangent plane.

Take, secondly, $$z = x^3.$$

Here $r = s = t = 0$. Since z is positive or negative according as x is positive or negative, the surface lies on both sides of the tangent plane.

Finally, consider $$z = x^2 - 3\,xy^2 + 2\,y^4.$$

If we follow the procedure used in Case II and place

$$x = \rho \cos\theta,$$
$$y = \rho \sin\theta,$$

we have $$z = \rho^2 - \rho^3(-3\cos\theta\sin^2\theta + 2\sin^4\theta). \qquad (13)$$

For any given value of θ the value of ρ may be taken so small that the magnitude of the first term exceeds numerically the value of the second term. In fact, we have only to take

$$\rho < \frac{1}{|-3\cos\theta\sin^2\theta + 2\sin^4\theta|}.$$

Consequently for any value of θ the value of z in (13) is positive for sufficiently small values of ρ. This may easily be mistaken for a proof that the surface is always above the xy-plane in the neighborhood of the origin. That this is not so may be seen by writing the equation of the surface in the form

$$z = (x - 2\,y^2)(x - y^2)$$

and drawing the curves (Fig. 39)

$$x - 2\,y^2 = 0 \quad \text{and} \quad x - y^2 = 0.$$

Then it is easily seen that for points in the regions marked + in the figure z is positive and for points marked − in the figure z is negative. On the other hand, any straight line through O crosses a + region before reaching the origin.

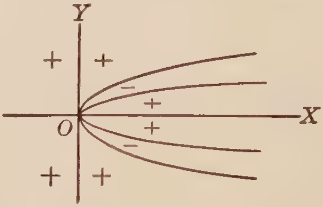

FIG. 39

If $rt - s^2 = 0$, the section of the surface made by a plane parallel to the tangent plane consists approximately of two parallel lines.

These being a special case of a parabola, the point is called a *parabolic* point.

50. Maxima and minima. The function $f(x, y)$ has a maximum value for $x = a$, $y = b$ if

$$f(a + h, b + k) > f(a, b) \tag{1}$$

for all values of h and k sufficiently small. Similarly, $f(x, y)$ has a minimum value for $x = a$, $y = b$ if

$$f(a + h, b + k) < f(a, b) \tag{2}$$

for all values of h and k sufficiently small. If we represent the function graphically by the surface

$$z = f(x, y), \tag{3}$$

we may at once apply the results of the previous section. In the first place it is evident that if z has a maximum or a minimum value c when $x = a$, $y = b$, the tangent plane of the surface must be parallel to the XOY plane. Hence it is necessary that

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0. \tag{4}$$

Further, it appears that if the point (a, b, c) is an elliptic point, z has a maximum or a minimum value according as the surface is below or above its tangent plane; if (a, b, c) is a hyperbolic point,

z has neither a maximum nor a minimum value. If (a, b, c) is a parabolic point, the question is doubtful. We may accordingly make the following statement:

In order that $f(x, y)$ *should have a maximum or a minimum value for* $x = a$, $y = b$, *it is necessary that for these values*

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

If, in addition, $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x\, \partial y}\right)^2 > 0$, $f(x, y)$ *has a maximum value when* $\dfrac{\partial^2 f}{\partial x^2}$ *and* $\dfrac{\partial^2 f}{\partial y^2} < 0$ *and a minimum value when* $\dfrac{\partial^2 f}{\partial x^2}$ *and* $\dfrac{\partial^2 f}{\partial y^2} > 0$.

If $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x\, \partial y}\right)^2 < 0$, *then* $f(x, y)$ *has certainly neither a maximum nor a minimum.*

If $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x\, \partial y}\right)^2 = 0$, *the matter is doubtful.*

Suppose, now, that we have a function of any number of variables

$$f(x, y, z, \cdots).$$

The geometric interpretation is now inconvenient even with the assumption of a space of four or more dimensions. Moreover, the necessary and sufficient conditions for a maximum or a minimum value are complicated. It is easy, however, to give necessary conditions. For if $f(x, y, z, \cdots)$ is to be a maximum no matter how $x, y, z, \cdots$ vary, it must be a maximum when one alone of these quantities varies. But the necessary condition that a function of a single variable should have a maximum or a minimum value is that its derivative should be zero. This is a well-known theorem of the elementary calculus and has been essentially proved in this text in § 5. Hence the *necessary* conditions that $f(x, y, z, \cdots)$ should have a maximum or a minimum value are

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0, \cdots. \tag{5}$$

In applied problems it is usually sufficient to solve equations (5) and then determine from the nature of the problem whether the solution gives a maximum or a minimum value of f, or neither.

51. Curves. We have already noted that the equations

$$\begin{aligned} x &= f_1(t), \\ y &= f_2(t), \\ z &= f_3(t), \end{aligned} \tag{1}$$

where t is an arbitrary variable, define a one-dimensional extent of points (x, y, z) which by definition form a curve. We shall assume that the functions involved in (1) are continuous and have derivatives.

The direction of the curve and the direction of the tangent line at any point have been shown to be $dx : dy : dz$. Hence the equations of the tangent line at (x, y, z) are

$$\frac{x - x_1}{dx} = \frac{y - y_1}{dy} = \frac{z - z_1}{dy}. \tag{2}$$

It is customary to speak of (x, y, z) and $(x + dx, y + dy, z + dz)$ as consecutive points on the curve. This is the language of infinitesimals, since, strictly speaking, the point $(x + dx, y + dy, z + dz)$ is on the tangent line and not on the curve.

Associated with the curve at each point is a definite plane called the osculating plane. This we may conveniently obtain through the notion of three consecutive points on a curve. Let (x_1, y_1, z_1) be a point of the curve corresponding to $t = t_1$, and let (x, y, z) be a point corresponding to $t = t_1 + h$. Then, by Taylor's series,

$$x = x_1 + hf_1'(t_1) + \frac{h^2}{2!} f_1''(t_1) + \cdots. \tag{3}$$

In (3) take $h = dx$. Then if we regard only infinitesimals of the first order, we have, from (3),

$$x = x_1 + dx,$$

and if we consider infinitesimals of the second order, we have

$$x = x_1 + dx + \tfrac{1}{2}\, d^2x.$$

Treating y and z in the same way, we have three points, namely, $P(x_1,\ y_1,\ z_1)$, $Q(x_1 + dx,\ y_1 + dy,\ z_1 + dz)$, $R(x_1 + dx + \tfrac{1}{2}\, d^2x,$ $y_1 + dy + \tfrac{1}{2}\, d^2y,\ z_1 + dz + \tfrac{1}{2}\, d^2z)$, each of which lies on the curve, except for infinitesimals of a certain order, and which we call consecutive points of the curve.

Any plane through P has the equation (§ 48)

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0. \tag{4}$$

If, in addition, it passes through Q and R, we have

$$\begin{aligned} A\,dx + B\,dy + C\,dz &= 0, \\ A\,d^2x + B\,d^2y + C\,d^2z &= 0. \end{aligned} \tag{5}$$

If $A:B:C$ are found from (5) and substituted in (4), we have

$$(dy\,d^2z - d^2y\,dz)(x - x_1) + (dz\,d^2x - d^2z\,dx)(y - y_1) + (dx\,d^2y - d^2x\,dy)(z - z_1) = 0, \tag{6}$$

or, in determinant form,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ dx & dy & dz \\ d^2x & d^2y & d^2z \end{vmatrix} = 0, \tag{7}$$

as the equation of the osculating plane.

The length of the curve s may be taken as the independent variable in the defining equations (1). For s is defined as a function of t by the integral

$$s = \int_{t_0}^{t} \sqrt{dx^2 + dy^2 + dz^2},$$

and, conversely, t is a function of s and may be replaced by s. When that is done we shall write

$$x' = \frac{dx}{ds}, \quad y' = \frac{dy}{ds}, \quad z' = \frac{dz}{ds}, \tag{8}$$

$$x'' = \frac{d^2x}{ds^2}, \quad y'' = \frac{d^2y}{ds^2}, \quad z'' = \frac{d^2z}{ds^2}. \tag{9}$$

Then $$x'^2 + y'^2 + z'^2 = 1; \tag{10}$$

from which $$x'x'' + y'y'' + z'z'' = 0. \tag{11}$$

At any point P (Fig. 40) we have three mutually perpendicular lines of importance. The first is the tangent line PT the direction cosines of which are

$$l_1 = x', \quad m_1 = y', \quad n_1 = z'. \tag{12}$$

The second is the line PB normal to the osculating plane. It is called the *binormal* to the curve. By (6), its direction cosines are given by

$$l_2 : m_2 : n_2 = y'z'' - y''z' : z'x'' - z''x' : x'y'' - x''y'. \tag{13}$$

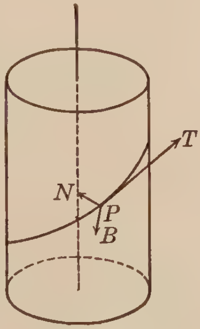

FIG. 40

C

If we form the identity

$$(y'z''-y''z')^2+(z'x''-z''x')^2+(x'y''-x''y')^2$$
$$=(x'^2+y'^2+z'^2)(x''^2+y''^2+z''^2)-(x'x''+y'y''+z'z'')^2$$

and apply (10) and (11), we have, from (13),

$$l_2=\frac{y'z''-y''z'}{\sqrt{x''^2+y''^2+z''^2}},$$
$$m_2=\frac{z'x''-z''x'}{\sqrt{x''^2+y''^2+z''^2}}, \qquad (14)$$
$$n_2=\frac{x'y''-x''y'}{\sqrt{x''^2+y''^2+z''^2}}.$$

The third line is the line PN lying in the osculating plane and perpendicular to the tangent line. It is called the *principal normal.* If l_3, m_3, n_3 are its direction cosines, then, since it is perpendicular to both PT and PB, we have

$$l_3x'+m_3y'+n_3z'=0,$$
$$l_3(y'z''-y''z')+m_3(z'x''-z''x')+n_3(x'y''-x''y')=0.$$

If the solution of these equations is simplified by the aid of (10) and (11), we have

$$l_3:m_3:n_3=x'':y'':z''; \qquad (15)$$

whence

$$l_3=\frac{x''}{\sqrt{x''^2+y''^2+z''^2}},$$
$$m_3=\frac{y''}{\sqrt{x''^2+y''^2+z''^2}}, \qquad (16)$$
$$n_3=\frac{z''}{\sqrt{x''^2+y''^2+z''^2}}.$$

The lines PB and PN determine a plane, the normal plane. It is obvious that any line in this plane is a normal to the curve.

We have been handling curves as defined by equations of type (1). It is evident, however, that two equations of the form

$$f(x, y, z)=0,$$
$$F(x, y, z)=0, \qquad (17)$$

also define a curve as the locus of points the coördinates of which satisfy both equations. This may be looked at in two ways: In the first place, each of the equations (17) taken alone defines a surface, and the points common to both surfaces lie on their curve of intersection; on the other hand, equations (17) are, by the theory of implicit functions, equivalent to the equations

$$\begin{aligned} y &= \phi(x), \\ z &= \psi(x), \end{aligned} \tag{18}$$

or, what is the same thing, to the three equations

$$\begin{aligned} x &= t, \\ y &= \phi(t), \\ z &= \psi(t), \end{aligned} \tag{19}$$

which are of type (1). Conversely, any equations of type (1) are equivalent to two equations of type (17).

52. Curvature and torsion. As a point P moves along a curve the direction of the tangent line changes and the osculating plane changes its orientation. The change of direction gives rise to the idea of *curvature*, and the change in the osculating plane to the idea of *torsion*.

To determine these we begin by deriving an expression for the angle between a line and another line very near it. Let l, m, n be the direction cosines of a straight line, so that

$$l^2 + m^2 + n^2 = 1, \tag{1}$$

and let $l + \Delta l$, $m + \Delta m$, $n + \Delta n$ be the direction cosines of the line when slightly displaced, so that

$$(l + \Delta l)^2 + (m + \Delta m)^2 + (n + \Delta n)^2 = 1; \tag{2}$$

whence, from (1),

$$2\,l\,\Delta l + 2\,m\,\Delta m + 2\,n\,\Delta n + (\Delta l)^2 + (\Delta m)^2 + (\Delta n)^2 = 0. \tag{3}$$

By (4), § 45, if $\Delta\theta$ is the angle between these two lines,

$$\begin{aligned} \cos\Delta\theta &= l^2 + l\,\Delta l + m^2 + m\,\Delta m + n^2 + n\,\Delta n \\ &= 1 - \tfrac{1}{2}[(\Delta l)^2 + (\Delta m)^2 + (\Delta n)^2]. \end{aligned} \tag{4}$$

But $\cos\Delta\theta = 1 - \frac{1}{2}(\Delta\theta)^2$, except for infinitesimals of higher order, and hence, except for infinitesimals of higher order,

$$(\Delta\theta)^2 = (\Delta l)^2 + (\Delta m)^2 + (\Delta n)^2.$$

Hence if l, m, n are functions of an independent variable t and we divide by Δt and pass to the limit, we have

$$\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{dl}{dt}\right)^2 + \left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2,$$

or, in differential form,

$$d\theta^2 = dl^2 + dm^2 + dn^2. \tag{5}$$

The *curvature* of a curve may be defined as the rate of change of the direction of a curve with respect to its length. More precisely, if $\Delta\theta$ is the angle between two tangents at points differing by Δs, then

$$\text{curvature} = \operatorname{Lim} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}. \tag{6}$$

This may be computed from (5) by the aid of (12), § 51, and we have

$$\text{curvature} = \sqrt{x''^2 + y''^2 + z''^2}.$$

The radius of curvature ρ is the reciprocal of the curvature; whence

$$\rho = \frac{1}{\sqrt{x''^2 + y''^2 + z''^2}}. \tag{7}$$

Using this result in (14) and (16), § 51, we have

$$\begin{aligned} l_2 &= \rho(y'z'' - y''z'), \quad m_2 = \rho(z'x'' - z''x'), \\ n_2 &= \rho(x'y'' - x''y'), \end{aligned} \tag{8}$$

and

$$l_3 = \rho x'', \quad m_3 = \rho y'', \quad n_3 = \rho z'', \tag{9}$$

and from (12), § 51, and (9), just found,

$$l_1' = \frac{l_3}{\rho}, \quad m_1' = \frac{m_3}{\rho}, \quad n_1' = \frac{n_3}{\rho}. \tag{10}$$

The *torsion* may be defined roughly as the rate of change of the position of the osculating plane with respect to the length of the curve. More precisely, if $\Delta\theta$ is the angle between two binormals, at points differing by Δs, then

$$\text{torsion} = \operatorname{Lim} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}.$$

The radius of torsion τ is defined as the reciprocal of the torsion, so that, by (5),

$$\frac{1}{\tau} = \sqrt{\left(\frac{dl_2}{ds}\right)^2 + \left(\frac{dm_2}{ds}\right)^2 + \left(\frac{dn_2}{ds}\right)^2}. \tag{11}$$

The direct calculation of this expression is tedious. We shall proceed indirectly as follows: The direction cosines of tangent, binormal, and principal normal satisfy the six equations

$$l_1^2 + m_1^2 + n_1^2 = 1, \tag{12}$$

$$l_2^2 + m_2^2 + n_2^2 = 1, \tag{13}$$

$$l_3^2 + m_3^2 + n_3^2 = 1, \tag{14}$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0, \tag{15}$$

$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0, \tag{16}$$

$$l_3 l_1 + m_3 m_1 + n_3 n_1 = 0, \tag{17}$$

each of the first three being the fundamental relation for direction cosines, and each of the last three the condition for perpendicularity.

Take (13) and (15) and differentiate with respect to s. We have

$$l_2'l_2 + m_2'm_2 + n_2'n_2 = 0, \tag{18}$$

$$l_2'l_1 + m_2'm_1 + n_2'n_1 = 0, \tag{19}$$

where equation (19) has been simplified by the aid of (10) and (16).

From (18) and (19) we have

$$l_2' : m_2' : n_2' = m_2n_1 - m_1n_2 : n_2l_1 - n_1l_2 : l_2m_1 - l_1m_2. \tag{20}$$

But from (16) and (17) we also get

$$l_3 : m_3 : n_3 = m_2n_1 - m_1n_2 : n_2l_1 - n_1l_2 : l_2m_1 - l_1m_2. \tag{21}$$

Hence

$$l_2' : m_2' : n_2' = l_3 : m_3 : n_3, \tag{22}$$

and therefore, by (11),

$$l_3 = \tau l_2', \quad m_3 = \tau m_2', \quad n_3 = \tau n_2'. \tag{23}$$

Take now equations (8) and differentiate with respect to s, paying attention to (23). We have

$$\begin{aligned} \frac{l_3}{\tau} &= \rho'(y'z'' - y''z') + \rho(y'z''' - y'''z') \\ &= \frac{\rho'}{\rho} l_2 + \rho(y'z''' - y'''z'), \\ \frac{m_3}{\tau} &= \frac{\rho'}{\rho} m_2 + \rho(z'x''' - z'''x'), \\ \frac{n_3}{\tau} &= \frac{\rho'}{\rho} n_2 + \rho(x'y''' - x'''y'). \end{aligned}$$

Multiply these equations in order by l_3, m_3, n_3, respectively, add, and reduce by (14), (16), and (9). We have

$$\begin{aligned} \frac{1}{\tau} &= \rho^2[x''(y'z''' - y'''z') + y''(z'x''' - z'''x') \\ &\qquad\qquad + z''(x'y''' - x'''y')] \\ &= -\rho^2 \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}. \end{aligned} \tag{24}$$

We have come out apparently with a negative sign, but as the sign of the determinant is not given, the sign of the torsion is not apparent from (24). As a matter of fact, it is possible so to determine the positive directions of the three principal lines that the sign of the torsion determines whether the osculating plane has a right-hand turning or a left-hand turning as it progresses along the curve, but this is a matter of detail into which we shall not go.

53. Curvilinear coördinates. The three equations

$$\begin{aligned} x &= f_1(u, v),\\ y &= f_2(u, v),\\ z &= f_3(u, v), \end{aligned} \tag{1}$$

where u and v are independent variables, in general define a surface. For unless all three of the Jacobians $J\left(\frac{x, y}{u, v}\right)$, $J\left(\frac{y, z}{u, v}\right)$, and $J\left(\frac{z, x}{u, v}\right)$ vanish, two of the equations may be solved for u and v, and the result substituted in the remaining equation. There results an equation of the form

$$F(x, y, z) = 0. \tag{2}$$

A particular form of equations (1) is

$$\begin{aligned} x &= u,\\ y &= v,\\ z &= f(u, v), \end{aligned} \tag{3}$$

which is obviously equivalent to

$$z = f(x, y), \tag{4}$$

already discussed.

If in (1) we place $v = c$, we have a curve lying on the surface, since u is now the only variable. Similarly, $u =$ constant gives a curve lying on the surface. The surface is then covered by two families of curves. To each point P (Fig. 41) correspond two values of u and v, and (u, v) are curvilinear coördinates on the surface.

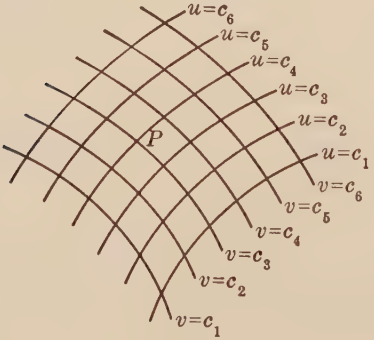

FIG. 41

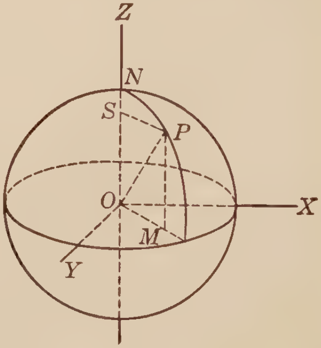

FIG. 42

Consider, for example, a sphere with center at O and radius a (Fig. 42). Let the axis of z intersect the sphere at N and let P be any point of the sphere. From P draw the line PS perpendicular to ON, the line PM perpendicular to the plane XOY, and the line $OP = a$. Let (r, θ) be the polar coördinates of M on the plane XOY; that is, $r = OM$ and $\theta = XOM$. Let the angle POS be called ϕ.

Then
$$OM = a \sin \phi,$$
$$x = OM \cos \theta = a \cos \theta \sin \phi,$$
$$y = OM \sin \theta = a \sin \theta \sin \phi,$$
$$z = a \cos \phi.$$

The angles (θ, ϕ) are then curvilinear coördinates on the sphere. The curves $\theta =$ constant are great circles through N, meridians. The lines $\phi =$ constant are small circles parallel to XOY, circles of latitude. In fact, θ and ϕ are precisely analogous to the longitude and co-latitude of points on the earth's surface.

If in the equation for the element of arc in space,

$$ds = dx^2 + dy^2 + dz^2,$$

we substitute the values of dx, dy, dz taken from (1), we find, as the element of arc on the surface,

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2, \tag{5}$$

where
$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2,$$
$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v},$$
$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

If $du : dv$ and $\delta u : \delta v$ are two directions on the surface corresponding to $dx : dy : dz$ and $\delta x : \delta y : \delta z$, respectively, in space, and θ is the angle between these directions, then, by (4), § 45,

$$\begin{aligned}\cos \theta &= \frac{dx\,\delta x + dy\,\delta y + dz\,\delta z}{ds\,\delta s}\\ &= \frac{E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v}{\sqrt{E\,du^2 + 2\,F\,du\,dv + G\,dv^2}\,\sqrt{E\,\delta u^2 + 2\,F\,\delta u\,\delta v + G\,\delta v^2}}.\end{aligned} \tag{6}$$

In particular, let ω be the angle between the coördinate curves $u = c$ (for which $du = 0$) and $v = c$ (for which $\delta v = 0$). Then (6) gives

$$\cos\omega = \frac{F\,\delta u\,dv}{\sqrt{G\,dv^2}\,\sqrt{E\,\delta u^2}} = \frac{F}{\sqrt{EG}}; \tag{7}$$

whence

$$\sin\omega = \frac{\sqrt{EG - F^2}}{\sqrt{EG}}. \tag{8}$$

From (7) it follows that *the necessary and sufficient condition that the coördinate system be orthogonal is that* $F = 0$.

Consider the infinitesimal figure $PQRS$ (Fig. 43) bounded by four coördinate lines. By (5),

$$PQ = \sqrt{E}\,du, \quad PS = \sqrt{G}\,dv.$$

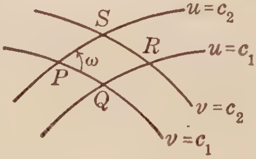

FIG. 43

Treating this figure as equivalent to a parallelogram, except for infinitesimals of higher order than du or dv, we have

$$\text{Area } PQRS = PQ \cdot PR \cdot \sin\omega = \sqrt{EG - F^2}\,du\,dv.$$

This we call the element of area dS and write

$$dS = \sqrt{EG - F^2}\,du\,dv. \tag{9}$$

A special case of (1) is obtained when we have

$$\begin{aligned} x &= f(u, v), \\ y &= f(u, v), \\ z &= 0, \end{aligned} \tag{10}$$

where u and v are curvilinear coördinates in the plane. All results hold. In particular, if we place $u = r$, $v = \theta$, and write

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned}$$

we have the usual polar coördinates, and (5) becomes

$$ds = dr^2 + r^2\,d\theta^2$$

and (9) becomes

$$dS = r\,dr\,d\theta.$$

Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction cosines of the normal to (1) at a point (u, v). Then, since the normal is perpendicular both to the curve $u = \text{constant}$ and to the curve $c = \text{constant}$, we have, by (5), § 45,

$$\frac{\partial x}{\partial u}\cos\alpha + \frac{\partial y}{\partial u}\cos\beta + \frac{\partial z}{\partial u}\cos\gamma = 0,$$
$$\frac{\partial x}{\partial v}\cos\alpha + \frac{\partial y}{\partial v}\cos\beta + \frac{\partial z}{\partial v}\cos\gamma = 0;$$

whence $\cos\alpha : \cos\beta : \cos\gamma = J\left(\frac{y,\ z}{u,\ v}\right) : J\left(\frac{z,\ x}{u,\ v}\right) : J\left(\frac{x,\ y}{u,\ v}\right).$

The student may verify by direct expansion that

$$\left[J\left(\frac{y,\ z}{u,\ v}\right)\right]^2 + \left[J\left(\frac{z,\ x}{u,\ v}\right)\right]^2 + \left[J\left(\frac{x,\ y}{u,\ v}\right)\right]^2 = EG - F^2.$$

Hence we have

$$\begin{aligned} \cos\alpha &= \frac{1}{\sqrt{EG - F^2}}\,J\left(\frac{y,\ z}{u,\ v}\right), \\ \cos\beta &= \frac{1}{\sqrt{EG - F^2}}\,J\left(\frac{z,\ x}{u,\ v}\right), \\ \cos\gamma &= \frac{1}{\sqrt{EG - F^2}}\,J\left(\frac{x,\ y}{u,\ v}\right). \end{aligned} \tag{11}$$

Again, let us place

$$\begin{aligned} x &= f_1(u,\ v,\ w), \\ y &= f_2(u,\ v,\ w), \\ z &= f_3(u,\ v,\ w), \end{aligned} \tag{12}$$

where u, v, and w are independent variables and where

$$J\left(\frac{x,\ y,\ z}{u,\ v,\ w}\right) \neq 0. \tag{13}$$

Then equations (12) can be solved for

$$\begin{aligned} u &= \phi_1(x,\ y,\ z), \\ v &= \phi_2(x,\ y,\ z), \\ w &= \phi_3(x,\ y,\ z). \end{aligned} \tag{14}$$

Then we have three families of surfaces $u = c_1$, $v = c_2$, $w = c_3$, the intersection of which determines a point with Cartesian coördinates $(x,\ y,\ z)$ or curvilinear coördinates $(u,\ v,\ w)$. The planes $x = c_1$, $y = c_2$, $z = c_3$, constructed for varying values of c_1, c_2, c_3, divide space into rectangular parallelepipeds. In the same manner, except for exceptional points, the space is divided into six-faced cells by the surfaces $u = c_1$, $v = c_2$, $w = c_3$ constructed for varying values of c_1, c_2, c_3.

The element of arc in curvilinear coördinates may be found by obtaining dx, dy, dz from (12) and substituting in

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{15}$$

but we have no need of writing out the general form which results.

To find the element of volume we begin by drawing from $P(x, y, z)$ (Fig. 44) three straight lines to $Q(x + dx, y + dy, z + dz)$, $R(x + \delta x, y + \delta y, z + \delta z)$, and $S(x + \Delta x, y + \Delta y, z + \Delta z)$ and constructing on these as edges a parallelepiped. Then, if θ is the angle between PQ and PR, the area of PRQ is $PQ \cdot PR \sin\theta$; and if ϕ is the angle between PS and the normal to the plane PRQ, the length of the perpendicular from S to the plane PRQ is $PS\cos\phi$. Hence the volume of the parallelepiped is

$$PQ \cdot PR \cdot PS \sin\theta\cos\phi.$$

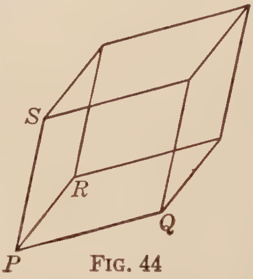

FIG. 44

The value of this may be worked out by the formulas of § 45 and found to be

$$\pm(dy\,\delta z - \delta y\,dz)\Delta x \pm (dz\,\delta x - \delta z\,dx)\Delta y \pm (dx\,\delta y - \delta x\,dy)\Delta z,$$

or, in determinant form,

$$\pm\begin{vmatrix} dx & dy & dz \\ \delta x & \delta y & \delta z \\ \Delta x & \Delta y & \Delta z \end{vmatrix}, \tag{16}$$

where the double sign is to be so chosen as to make the expression positive. In this formula the lengths of the sides may be as large as we please, and the formula is exact. Let us now, in place of the straight-line figure, place a six-sided figure bounded by surfaces of our curvilinear-coördinate system, formed by taking four points $P(u, v, w)$, $Q(u + du, v, w)$, $R(u, v + dv, w)$, and $S(u, v, w + dw)$ and passing coördinate surfaces through these.

Then, for the point Q,

$$dx = \frac{\partial x}{\partial u}du, \quad dy = \frac{\partial y}{\partial u}du, \quad dz = \frac{\partial z}{\partial u}du;$$

for R,

$$\delta x = \frac{\partial x}{\partial v}dv, \quad \delta y = \frac{\partial y}{\partial v}dv, \quad \delta z = \frac{\partial z}{\partial v}dv;$$

and for S,

$$\Delta x = \frac{\partial x}{\partial w}dw, \quad \Delta y = \frac{\partial y}{\partial w}dw, \quad \Delta z = \frac{\partial z}{\partial w}dw,$$

and we take the definition of the element of volume to be that obtained from (16) by substituting these values. We have

$$dV = \pm \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix} du\, dv\, dw \tag{17}$$

$$= \pm J\left(\frac{x,\ y,\ z}{u,\ v,\ w}\right) du\, dv\, dw.$$

This definition seems to be based on the assumption that the volume of the curved figure differs from that of the straight-line figure by an infinitesimal of higher order than the one taken. Its real justification lies in the fact that it is possible to prove with perfect rigor that the volume of a finite solid computed by it is a number independent of the coördinate system used, where it is to be noted that

$$dV = dx\, dy\, dz \tag{18}$$

is a special case of (17) obtained by placing $u = x,\ v = y,\ w = z$.

Two systems of curvilinear coördinates are in common use. The first are the cylindrical coördinates $(r,\ \theta,\ z)$ (Fig. 45), where

$$\begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta, \\ z &= z. \end{aligned} \tag{19}$$

These are equivalent to taking polar coördinates on the plane XOY and leaving the z coördinate unchanged. The coördinate surfaces are $r = c$, concentric cylinders with OZ as axis; $\theta = c$, planes through OZ; and $z = c$, planes perpendicular to OZ. In cylindrical coördinates we have

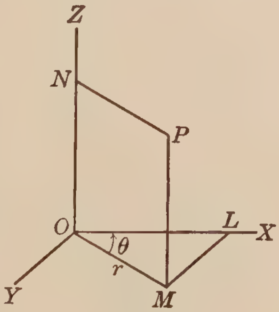

FIG. 45

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2 \tag{20}$$

and

$$dV = r\, d\theta\, dr\, dz. \tag{21}$$

The second set of curvilinear coördinates in common use are polar, or spherical, coördinates (r, θ, ϕ) (Fig. 46), by which

$$\begin{aligned} x &= r \sin\phi \cos\theta, \\ y &= r \sin\phi \sin\theta, \qquad (22) \\ z &= r \cos\phi. \end{aligned}$$

The coördinate surfaces are $r = c$, spheres with center at O; $\theta = c$, planes through OZ; and $\phi = c$, circular cones with OZ as an axis. In polar coördinates we have

$$ds^2 = dr^2 + r^2 \sin^2\phi \, d\theta^2 + r^2 \, d\phi^2 \qquad (23)$$

and

$$dV = r^2 \sin\phi \, dr \, d\theta \, d\phi. \qquad (24)$$

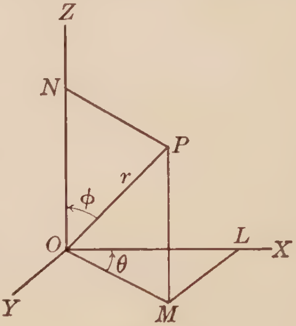

FIG. 46

EXERCISES

1. If $A_1 : B_1 : C_1$ and $A_2 : B_2 : C_2$ fix the directions of two straight lines and $A_3 : B_3 : C_3$ fix the direction of a line perpendicular to them, prove that

$$A_3 : B_3 : C_3 = B_1C_2 - B_2C_1 : C_1A_2 - C_2A_1 : A_1B_2 - A_2B_1.$$

2. Find the direction of the curve $x = e^t$, $y = e^{-t}$, $z = t\sqrt{2}$ at the point for which $t = 0$.

3. Show that the curve $x = a \cos t$, $y = a \sin t$, $z = kt$ makes a constant angle with the direction parallel to OZ.

4. Show that the locus of points whose coördinates satisfy simultaneously the equations $f(x, y, z) = 0$, $g(x, y, z) = 0$, has in general a definite direction at each point of space, and determine the direction.

5. Find the tangent plane to the paraboloid $z = ax^2 + by^2$ at the point (x_1, y_1, z_1).

6. Find the tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (x_1, y_1, z_1).

7. Prove that the plane $lx + my + nz = p$ is tangent to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ if $p = \sqrt{a^2l^2 + b^2m^2 + c^2n^2}$.

8. Prove that the plane $lx + my + nz = p$ is tangent to the paraboloid $ax^2 + by^2 = z$ if $p = -\frac{bl^2 + am^2}{4\,abn}$.

9. Find the cosine of the angle between the normal to an ellipsoid and the straight line drawn from the center to the point of contact, and prove that it is equal to $\frac{p}{r}$, where p is the distance of the tangent plane from the center and r is the distance of the point of contact from the center.

10. Find the angle between the line drawn from the origin to the point (a, a, a) of the surface $xyz = a^3$ and the normal of the surface at the point.

11. Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = a^2$ and $(x - b)^2 + y^2 + z^2 = c^2$.

12. Derive the condition that two surfaces $f(x, y, z) = 0$ and $\phi(x, y, z) = 0$ intersect at right angles.

13. Find the point in the plane $ax + by + cz + d = 0$ which is nearest the origin.

14. Find the points on the surface $xyz = a^3$ which are nearest the origin.

15. Find a point in a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

16. Of all rectangular parallelepipeds inscribed in an ellipsoid, find that which has the greatest volume.

17. Find the point inside a plane triangle from which the sum of the squares of the perpendiculars to the three sides is a minimum. (Express the answer in terms of K, the area of the triangle; a, b, c, the lengths of the three sides; and x, y, z, the three perpendiculars on the sides.)

18. Show that the necessary conditions for the maximum and minimum values of $f(x, y)$, where x and y are connected by an equation $F(x, y) = 0$, is that x and y should satisfy the two equations

$$F(x, y) = 0,$$
$$\frac{\partial f}{\partial x}\frac{\partial F}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial F}{\partial x} = 0.$$

19. Find the lengths of the shortest and longest lines from the origin to the conic $ax^2 + 2\,hxy + by^2 = c$. Find also the direction of these lines (axes of the conic).

20. Determine x, y, and z so that $x^p y^q z^r$ shall be a maximum if $x + y + z = N$, where p, q, r, and N are constants.

21. Show that the necessary conditions for a maximum or a minimum value of $f(x, y, z)$, where x, y, and z are connected by the condition $F(x, y, z) = 0$, is that x, y, and z should satisfy the three equations

$$F(x, y, z) = 0,$$
$$\frac{\partial f}{\partial x}\frac{\partial F}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial F}{\partial x} = 0,$$
$$\frac{\partial f}{\partial y}\frac{\partial F}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial F}{\partial y} = 0.$$

22. Prove that any two linear equations

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$

define a straight line, and show that its direction is given by

$$B_1C_2 - B_2C_1 : C_1A_2 - C_2A_1 : A_1B_2 - A_2B_1.$$

23. Consider the helix, or screw curve,

$$x = a \cos \theta,$$
$$y = a \sin \theta,$$
$$z = k\theta.$$

Show that it winds around a cylinder, and find the equations of its three principal lines and of its osculating plane.

24. Find the angle at which the helix (Ex. 23) cuts the elements of the cylinder on which it lies.

25. Consider the conical helix

$$x = t \cos t,$$
$$y = t \sin t,$$
$$z = kt.$$

Show that it winds around a cone, and find the equations of its three principal lines and of its osculating plane.

26. Find the angle at which the conical helix (Ex. 25) cuts the elements of the cone on which it lies.

27. Show that if the osculating plane of a curve is the same at all points the curve lies in that plane, and conversely.

28. Find the radii of curvature and of torsion of the helix.

29. Find the radius of curvature of the conical helix at the origin.

In each of the following examples find the Cartesian equation of the surface, the nature of the coördinate curves, the element of arc, and the element of area:

30. $x = u \cos v,$
$y = u \sin v,$
$z = ku.$

31. $x = u \cos v,$
$y = u \sin v,$
$z = kv.$

32. $x = au \cos v,$
$y = bu \sin v,$
$z = cu.$

33. $x = a \cos v,$
$y = b \sin v,$
$z = u.$

34. $x = a \sin u \cos v,$
$y = b \sin u \sin v,$
$z = c \cos u.$

35. $x = u \cos v,$
$y = u \sin v,$
$z = u^2.$

36. Show that any surface of revolution may be given the equations

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = f(r),$$

where r is the distance from OZ of a point on the surface, and $z = f(x)$ is the curve revolved to form the surface. Find the coördinate curves and the elements of length and area.

37. Let the points (x, y) of a plane be made to correspond to the points (ϕ, θ) of a sphere of radius a by the equations

$$x = k\theta,$$
$$y = k\operatorname{sech}^{-1}(\sin\phi).$$

This is Mercator's projection, much used in map-making. Show that meridians of longitude and circles of latitude on the sphere become straight lines on the plane. Show that if ds is the element of arc on the plane and ds' the element of arc on the sphere, then

$$ds = \frac{k}{a\sin\phi}\,ds'.$$

Show that angles are preserved (that is, the angle between any two curves on the sphere is the same as the angle between their corresponding curves on the plane), but that distances are magnified in a variable manner, the magnification becoming greater the farther one goes from the equator.

38. Let a sphere be mapped on a plane by the equations

$$x = k\tan\frac{\phi}{2}\cos\theta.$$

$$y = k\tan\frac{\phi}{2}\sin\theta.$$

This is stereographic projection. Show that angles are preserved and distances magnified in a varying manner. Find the curves corresponding to meridians and circles of latitude.

39. A loxodrome is a curve which cuts meridians on a sphere at a constant angle. Show that it becomes a straight line in Mercator's projection (Ex. 37) and a logarithmic spiral in stereographic projection (Ex. 38).

40. Show as a generalization of Exs. 37–38 that if a surface with coördinates (u, v) is mapped upon a surface with coördinates (u', v') so that $ds^2 = \lambda\, ds'^2$, where λ is a function of u' and v', angles will be preserved.

CHAPTER VI

THE DEFINITE INTEGRAL

54. Definition. The concept of the definite integral is obtained as follows:

In the interval $a \leqq x \leqq b$ (Fig. 47) assume at pleasure n points $x_0 = a, x_1, x_2, x_3, \cdots, x_{n-1}$, where $x_{i+1} > x_i$, thus dividing (a, b) into n smaller intervals. In each of these intervals take a value of $x = \xi_i$, where $x_{i-1} \leqq \xi_i \leqq x_i$, and form the sum

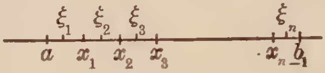

FIG. 47

$$\sum_{i=0}^{i=n-1} f(\xi_{i+1})(x_{i+1} - x_i) = f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(b - x_{n-1}). \quad (1)$$

Now let n increase indefinitely while each of the n intervals $x_{i+1} - x_i$ approaches zero. If the sum (1) approaches a limit which is independent of the choice of x_i or of ξ_i, that limit is called the definite integral of $f(x)$ between a and b and is denoted by

$$\int_a^b f(x)dx.$$

In the next section a proof of the existence of the limit will be given under certain conditions. Here it may be made graphically plausible that the limit exists if $f(x)$ is continuous and a and b finite. For if $f(x)$ is expressed by a graph, we have a figure like Fig. 48. The sum (1) represents the sum of the rectangles of the figure, and it seems obvious that the limit of the sum is the area bounded by the curve, the axis of x, and the ordinates $x = a$ and $x = b$.

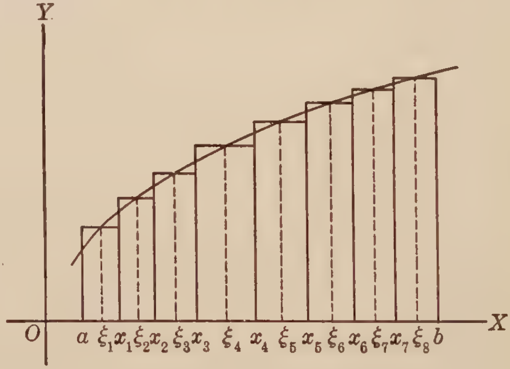

FIG. 48

Also, if $f(x)$ has a finite number of finite discontinuities, but a and b are finite, as in Fig. 49, the area and the integral seem to exist. The student may accept these graphical arguments or read the next section.

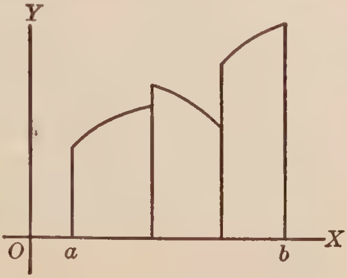

FIG. 49

55. Existence proof. Let $f(x)$ be a function which does not become infinite in the interval (a, b), M_{i+1} the largest value of $f(x)$ in the interval (x_i, x_{i+1}), and m_{i+1} the smallest value of $f(x)$ in the same interval. Form the two sums:

$$S = M_1(x_1 - a) + M_2(x_2 - x_1) + \cdots + M_n(b - x_{n-1}), \qquad (1)$$

$$s = m_1(x_1 - a) + m_2(x_2 - x_1) + \cdots + m_n(b - x_{n-1}). \qquad (2)$$

Then $$s < S.$$

Now let each of the divisions x_i, x_{i+1} be subdivided into smaller intervals and let the sums S' and s' be formed as before. We wish to show that

$$S' < S,$$
$$s' > s.$$

To show this suppose that the points $y_1, y_2, \cdots, y_k$ be chosen between x_i and x_{i+1}, as sketched in Fig. 50.

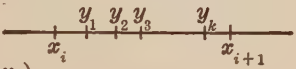

FIG. 50

Then, in place of the term $M_{i+1}(x_{i+1} - x_i)$ of (1), in S' there appear $k + 1$ terms,

$$M'_1(y_1 - x_i) + M'_2(y_2 - y_1) + \cdots + M'_{k+1}(x_{i+1} - y_k). \qquad (3)$$

But unless $f(x)$ is constant in the interval (x_i, x_{i+1}) some of the quantities M_p' are less than M_{i+1}, the points $y_1, y_2, \cdots, y_k$ being taken at pleasure. Hence the sum of the terms (3) is less than $M_{i+1}(x_{i+1} - x_i)$, and therefore $S' < S$. Similarly, $s' > s$. Hence as the subdivision is carried farther and farther, S constantly decreases and s constantly increases. There would be an exception only in the trivial case in which $f(x)$ is constant between $x = a$ and $x = b$.

We shall assume as evident that if a quantity is constantly decreasing, it either approaches a limit or eventually becomes less than any assigned negative quantity. But S is always greater than s. Therefore S approaches a limit. Similarly s, which constantly increases but remains smaller than S, approaches a limit.

It remains to show that under certain conditions the limits of S and s are the same.

From (1) and (2),

$$S - s = \sum (M_{i+1} - m_{i+1})(x_{i+1} - x_i).$$

Now if $f(x)$ is continuous in the interval (a, b), then from IV, § 2, we may take all the intervals $x_{i+1} - x_i$ so small that $M_{i+1} - m_{i+1} < \epsilon$ for all the intervals at the same time. Then

$$S - s < \epsilon \sum (x_{i+1} - x_i) = \epsilon(b - a);$$

whence $$\text{Lim } S = \text{Lim } s = I \tag{4}$$

if $b - a$ is finite.

We have proved the existence of the limit I for a particular original division of (a, b). To complete the proof we need to show that the limit is the same for any other subdivision. Let S and s be, as before, the sums (1) and (2), and let S'' and s'' be the sums similar to (1) and (2), with the points x_i replaced by other points x_i''. Then, by previous proof,

$$\text{Lim } S'' = \text{Lim } s'' = I''.$$

Finally, let S' and s' be the sums formed by considering both the x_i and the x_i' points. Then S' is a sum formed by subdividing both the intervals used in forming S and the intervals used in forming S''. Hence the limit of S' is at the same time the limit of S and the limit of S''; whence

$$I'' = I.$$

We have proved the existence of the limit of the sum (1) when $f(x)$ is continuous and the interval (a, b) is finite. Let us now suppose that $f(x)$ has a finite number k of points of finite discontinuity such as are pictured in Fig. 49.

Let the interval (a, b) be divided in any manner, and let the sum of the lengths of the intervals in which the points of discontinuity lie be l. If the intervals are small enough, only one point of discontinuity will lie in any one interval. In these k intervals let the difference between the largest and the smallest value of $f(x)$ be B. The sum of the lengths of the intervals in which there is no point of discontinuity is $b - a - l$, and the difference between M_{i+1} and m_{i+1} in each of these intervals may be made less than ϵ by taking the intervals sufficiently small.

Hence, using S and s in the same sense as before,

$$S - s < \epsilon(b - a - l) + Bl.$$

Now both ϵ and l approach zero as the division is made smaller, and therefore

$$\text{Lim } S = \text{Lim } s.$$

We have limited ourselves to a finite number of discontinuities. As a matter of fact, the reasoning applies to an infinite number provided $l \rightarrow 0$. A discussion of this case would necessitate a treatment of point sets, which we shall not give.

We have proved that S and s approach the same limit under the hypotheses made. It is obvious that the sum (1), § 54, is intermediate in value between S and s and approaches the same limit.

We have now the following theorem:

If $f(x)$ is a function continuous in the interval (a, b), with at most a finite number of finite discontinuities, the definite integral $\int_a^b f(x)dx$ exists.

56. Properties of definite integrals. As immediate consequences of the definition of the definite integral we have the following formulas:

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx, \tag{1}$$

$$\int_a^b [f_1(x) + f_2(x)]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx, \tag{2}$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx, \tag{3}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \tag{4}$$

$$\int_a^b f(x)dx = (b-a)f(\xi). \quad (a < \xi < b) \tag{5}$$

In proving (3) we have to notice that to interchange the limits a and b is to change the sign of each factor $x_{i+1} - x_i$ of (1), § 54. In (4) the order of magnitude of the quantities a, b, c is immaterial by virtue of (3). It is of course understood that $f(x)$ has the properties required in § 55.

To prove (5) we shall assume that $f(x)$ is continuous and not constant and shall let M be its maximum value and m its minimum value in the interval (a, b). If $f(\xi_{i+1})$ is replaced by M in each term of (1), § 54, the sum becomes $M(b-a)$, which is larger than the sum as written. Also, if $f(\xi_{i+1})$ is replaced by m, the sum becomes $m(b-a)$, which is evidently smaller than the sum as written. This is independent of the number of small intervals of (a, b). Hence

$$m(b-a) < \int_a^b f(x)dx < M(b-a);$$

whence

$$\int_a^b f(x)dx = \mu(b-a),$$

where μ is some number between m and M. But by II, § 2, since $f(x)$ is continuous in the interval (a, b), $f(x)$ takes the value μ for some value ξ of x in the interval. Hence (5) follows. Graphically this formula says that the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is equal to that of a rectangle with base $b - a$ and altitude equal to the height of some point of the curve (Fig. 51). This is obviously not necessarily true if $f(x)$ is discontinuous.

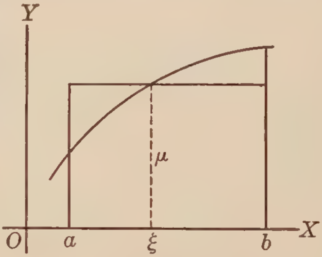

FIG. 51

57. Evaluation of a definite integral. Let $f(x)$ be a continuous function in the interval (a, b). Take x, any value in that interval. Then, since the integral is fully defined when the limits a and x are given, and that value depends upon the limits, we have, by the definition of a function,

$$\int_a^x f(x)dx = \phi(x).$$

Then

$$\begin{aligned}\phi(x+h) - \phi(x) &= \int_a^{x+h} f(x)dx - \int_a^x f(x)dx \\ &= \int_x^{x+h} f(x)dx \\ &= hf(\xi), \quad (x < \xi < x+h)\end{aligned}$$

where the transformations have been made by the formulas of the previous section.

Our result shows that $\phi(x)$ is a continuous function, since

$$\lim_{h \to 0} [\phi(x+h) - \phi(x)] = 0.$$

Also

$$\lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} = \lim_{\xi \to x} f(\xi) = f(x);$$

that is,

$$\frac{d}{dx}\phi(x) = f(x). \tag{1}$$

If, now, $F(x)$ is any function whose derivative is $f(x)$, then, by III, § 6, $\phi(x) = F(x) + C$, and we have

$$\int_a^x f(x)dx = F(x) + C.$$

If $x = a$, then, by the definition of the integral, the value of the integral is 0. Hence $C = -F(a)$. Using this value of C in our last formula and placing the upper limit x equal to b, we have finally

$$\int_a^b f(x)dx = F(b) - F(a). \qquad (2)$$

58. Simpson's rule. When the integral cannot be evaluated in elementary functions, recourse is sometimes had to approximate integration. The most obvious thing is to expand into a series and integrate term by term. From this method we may develop a rule known as *Simpson's rule.* Let

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + R, \qquad (1)$$

and let us integrate between a and $a + h$, omitting the remainder R. Then, approximately,

$$\int_a^{a+h} f(x)dx = a_0h + \frac{a_1h^2}{2} + \frac{a_2h^3}{3} + \frac{a_3h^4}{4}. \qquad (2)$$

Let y_1 be the value of $f(x)$ when $x = a$, y_2 the value of $f(x)$ when $x = a + \frac{h}{2}$, and y_3 the value of $f(x)$ when $x = a + h$ (Fig. 52). Then, approximately,

$$y_1 = a_0,$$
$$y_2 = a_0 + a_1\frac{h}{2} + \frac{a_2h^2}{4} + \frac{a_3h^3}{8},$$
$$y_3 = a_0 + a_1h + a_2h^2 + a_3h^3;$$

and from (2) it is easy to verify that, approximately,

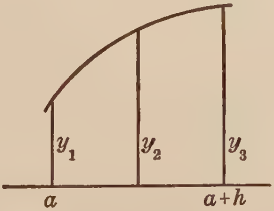

FIG. 52

$$\int_a^{a+h} f(x)dx = \frac{h}{6}(y_1 + 4y_2 + y_3). \qquad (3)$$

This is merely approximate, since we have omitted R in (1), but the error made is of the order of h^5 and is negligible for small values of h.

Consider now the integral

$$\int_a^b f(x)dx,$$

where b (Fig. 53) is too far from a for (3) to be applied to advantage. Let the interval (a, b) be divided into $2n$ equal parts $\frac{b-a}{2n}$, and take $h = \frac{b-a}{n} =$ two of such parts.

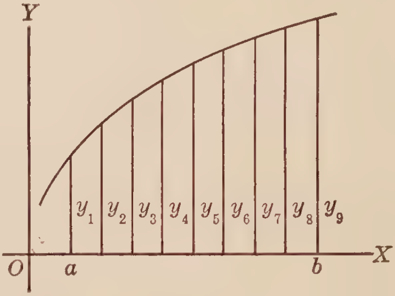

FIG. 53

The integral may be computed approximately by (3) for the intervals

$$(a,\ a+h),$$
$$(a+h,\ a+2h), \cdots,$$

and the results added. We obtain

$$\int_a^b f(x)dx = \frac{b-a}{6n}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \cdots + 4y_{2n} + y_{2n+1}). \tag{4}$$

This is Simpson's rule for the approximate value of an integral. It may be applied to computing an area where the ordinates may be measured from a carefully drawn diagram or computed from the equation of the curve.

59. Change of variables. Given

$$\int_a^b f(x)dx,$$

let it be required to place $x = \phi(t)$, where $x = a$ when $t = t_0$ and $x = b$ when $t = t_1$. A direct substitution gives

$$\int_{t_0}^{t_1} f[\phi(t)]\phi'(t)dt,$$

but this needs justification. For

$$\int_a^b f(x)dx = \text{Lim} \sum f(\xi_i)(x_{i+1} - x_i) = \text{Lim} \sum f(\xi_i)\Delta x_i,$$

and dx_i is the principal part of Δx_i. The principal part of Δx_i, however, may be substituted for dx_i by theorem II, § 12. The final work is therefore correct.

The proof of II, § 12, demands that the infinitesimals be all positive or all negative, but it is usually possible to split the interval (a, b) into regions in which the infinitesimals are always positive or always negative so as to apply the theorem.

60. Differentiation of a definite integral. The discussion of § 57 shows that the definite integral is a function of its upper limit b and of its lower limit a. Also, if $f(x)$ is a continuous function of x when $x = a$ or $x = b$, from (2), § 57,

$$\frac{\partial}{\partial b}\int_a^b f(x)dx = f(b), \quad \frac{\partial}{\partial a}\int_a^b f(x)dx = -f(a). \tag{1}$$

Suppose, now, that a and b are constant, but that $f(x)$ involves a parameter α which is constant in the integration but may vary to form different integrals; then, by definition of a function,

$$\int_a^b f(x, \alpha)dx = \phi(\alpha). \tag{2}$$

We shall show that in general $\phi(\alpha)$ may be differentiated by differentiating under the integral sign; thus,

$$\frac{d\phi}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx. \tag{3}$$

To prove this and, at the same time, to determine conditions under which the formula is true, we proceed as follows:

From (2) and the formulas of § 56 we have

$$\begin{aligned}\Delta\phi = \phi(\alpha + \Delta\alpha) - \phi(\alpha) &= \int_a^b f(x, \alpha + \Delta\alpha)dx - \int_a^b f(x, \alpha)dx \\ &= \int_a^b [f(x, \alpha + \Delta\alpha) - f(x, \alpha)]dx. \quad (4)\end{aligned}$$

Graphically $\phi(\alpha)$ is the area $ABDC$ (Fig. 54) and $\Delta\phi$ is the area $CDFE$. If $f(x, \alpha)$ is a continuous function of x and α when $a \leqq x \leqq b$ and α lies between two values, say α_0 and α_1, then, by § 30, we may take $\Delta\alpha$ so small that

$$|f(x, \alpha + \Delta\alpha) - f(x, \alpha)| < \epsilon$$

for all values of x in the interval (a, b). Graphically this means that the width of the strip $CDFE$ is less than ϵ for all points between A and B. Therefore, from (4),

$$|\Delta\phi| < \epsilon(b - a),$$

and $\phi(\alpha)$ is a continuous function.

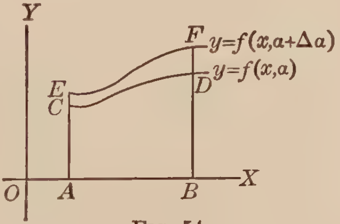

FIG. 54

From (4), $$\frac{\Delta\phi}{\Delta\alpha}=\int_a^b\frac{f(x,\,\alpha+\Delta\alpha)-f(x,\,\alpha)}{\Delta\alpha}\,dx. \tag{5}$$

Now if $\frac{\partial f}{\partial\alpha}$ exists and is continuous, then (5) is

$$\frac{\Delta\phi}{\Delta\alpha}=\int_a^b\frac{\partial f}{\partial\alpha}\,dx+\int_a^b\epsilon\,dx. \tag{6}$$

The last integral in (6) is less in absolute value than $\eta(b-a)$ if η is larger than any value of ϵ in the interval $(a,\,b)$. If $\frac{\partial f}{\partial\alpha}$ is continuous, the value of η may be made as small as we please by taking ϵ sufficiently small. Hence, taking the limit as $\Delta\alpha\rightarrow 0$ in (6), we have

$$\frac{d\phi}{d\alpha}=\int_a^b\frac{\partial f}{\partial\alpha}\,dx,$$

which is formula (3).

Now let us suppose that we have

$$\int_a^b f(x,\,\alpha)dx=\phi(\alpha), \tag{7}$$

where a and b are functions of α which take increments Δa and Δb, respectively, when α is increased by $\Delta\alpha$.

Then $$\begin{aligned}\phi(\alpha+\Delta\alpha)&=\int_{a+\Delta a}^{b+\Delta b}f(x,\alpha+\Delta\alpha)dx\\&=\int_{a+\Delta a}^{a}f(x,\,\alpha+\Delta\alpha)dx+\int_a^b f(x,\,\alpha+\Delta\alpha)dx\\&\qquad+\int_b^{b+\Delta b}f(x,\,\alpha+\Delta\alpha)dx,\end{aligned}$$

and $$\begin{aligned}\Delta\phi=\int_{a+\Delta a}^{a}f(x,\,\alpha+\Delta\alpha)dx&+\int_a^b[f(x,\,\alpha+\Delta\alpha)-f(x,\,\alpha)]dx\\&+\int_b^{b+\Delta b}f(x,\,\alpha+\Delta\alpha)dx.\end{aligned} \tag{8}$$

Graphically $\Delta\phi$ is represented in Fig. 55 by the unshaded border of the area denoted by $\phi(\alpha)$, and the three integrals in (8) give the areas of the strips *EAHG*, *CDIH*, and *BFJI* respectively. We may apply (5), § 56, to the first and last integrals of (8) and have

$$\begin{aligned}\Delta\phi=-\,\Delta af(\xi_1,\,\alpha+\Delta\alpha)&+\int_a^b[f(x,\,\alpha+\Delta\alpha)-f(x,\,\alpha)]dx\\&+\Delta bf(\xi_2,\,\alpha+\Delta\alpha).\end{aligned}$$

Dividing by $\Delta\alpha$, letting $\Delta\alpha \to 0$, noticing that $\xi_1 \to a$ and $\xi_2 \to b$, and using the result (3), we have

$$\frac{d\phi}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}\,dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}. \tag{9}$$

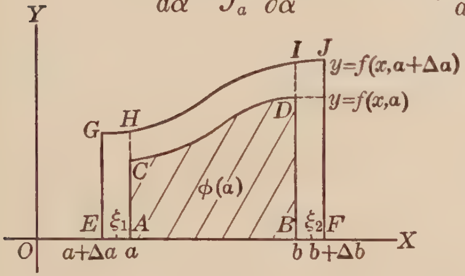

FIG. 55

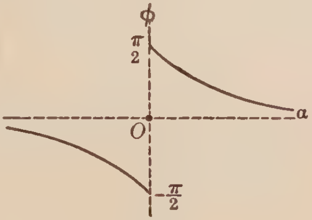

FIG. 56

As an illustration take

$$\phi(\alpha) = \int_0^1 \frac{\alpha}{x^2+\alpha^2}\,dx. \tag{10}$$

If $\qquad \alpha = 0, \qquad \phi(\alpha) = 0.$

If $\qquad \alpha \neq 0, \qquad \phi(\alpha) = \tan^{-1}\dfrac{1}{\alpha}.$

The function $\dfrac{\alpha}{x^2+\alpha^2}$ is not continuous at the point $x = 0$, $\alpha = 0$, and the function $\phi(\alpha)$ has a discontinuity when $\alpha = 0$. In fact, $\phi(\alpha)$ approaches $\pm\dfrac{\pi}{2}$ according as α is positive or negative. The graph of the function is shown in Fig. 56.

If we differentiate under the integral sign in (10), we have

$$\phi'(\alpha) = \int_0^1 \frac{x^2-\alpha^2}{(x^2+\alpha^2)^2}\,dx = \left[-\frac{x}{x^2+\alpha^2}\right]_0^1 = -\frac{1}{1+\alpha^2},$$

which is true for all values of α except $\alpha = 0$.

The principle of differentiating under the integral sign may sometimes be used to evaluate a definite integral. For example, take

$$\phi(\alpha) = \int_0^\pi \log(1 - 2\,\alpha\cos x + \alpha^2)dx. \tag{11}$$

$$\begin{aligned}\frac{d\phi}{d\alpha} &= \int_0^\pi \frac{-2\cos x + 2\,\alpha}{1 - 2\,\alpha\cos x + \alpha^2}\,dx\\ &= \frac{1}{\alpha}\int_0^\pi \left[1 - \frac{1-\alpha^2}{1 - 2\,\alpha\cos x + \alpha^2}\right]dx\\ &= \frac{\pi}{\alpha} - \frac{2}{\alpha}\left[\tan^{-1}\left(\frac{1+\alpha}{1-\alpha}\tan\frac{x}{2}\right)\right]_0^\pi.\end{aligned}$$

As x varies for 0 to π, $\frac{1+\alpha}{1-\alpha}\tan\frac{x}{2}$ varies through positive values for 0 to ∞ when $-1 < \alpha < 1$, and $\frac{1+\alpha}{1-\alpha}\tan\frac{x}{2}$ varies through negative values from 0 to $-\infty$ when $\alpha < -1$ or $\alpha > 1$.

Hence

$$\left[\tan^{-1}\left(\frac{1+\alpha}{1-\alpha}\tan\frac{x}{2}\right)\right]_0^\pi = \frac{\pi}{2} \quad \text{when} \quad -1 < \alpha < 1,$$

and $$\left[\tan^{-1}\left(\frac{1+\alpha}{1-\alpha}\tan\frac{x}{2}\right)\right]_0^\pi = -\frac{\pi}{2} \quad \text{when} \quad \alpha < -1 \text{ or } \alpha > 1.$$

Therefore

$$\frac{d\phi}{d\alpha} = 0 \quad \text{when} \quad -1 < \alpha < 1,$$

and $$\frac{d\phi}{d\alpha} = \frac{2\pi}{\alpha} \quad \text{when} \quad \alpha < -1 \text{ or } \alpha > 1;$$

whence $$\phi = C_1 \quad \text{when} \quad -1 < \alpha < 1,$$

and $$\phi = \pi \log \alpha^2 + C_2 \quad \text{when} \quad \alpha < -1 \text{ or } \alpha > 1.$$

We may determine C_1 by placing $\alpha = 0$ in (11). Then $C_1 = 0$.

Hence $$\phi = 0 \quad \text{when} \quad -1 < \alpha < 1. \tag{12}$$

To determine C_2 in the same manner we should need to substitute in (11) a value of α greater numerically than 1. This is not convenient. Instead, we will place

$\alpha = \frac{1}{\beta}$, where $-1 < \beta < 1$. Then

$$\begin{aligned}\phi(\alpha) &= \int_0^\pi [\log(1 - 2\beta\cos x + \beta^2) \\ &\qquad - \log\beta^2]dx \\ &= -\log\beta^2 \int_0^\pi dx \qquad \text{[by (12)]} \\ &= -\pi\log\beta^2 \\ &= \pi\log\alpha^2 \quad \text{when} \quad \alpha < -1 \\ &\qquad \text{or } \alpha > 1. \qquad (13)\end{aligned}$$

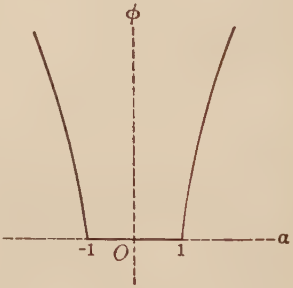

FIG. 57

Therefore $C_2 = 0$.

The definition of $\phi(\alpha)$ is now complete. Its graph is shown in Fig. 57.

The foregoing discussion does not apply when $\alpha = \pm 1$, since the conditions for differentiability are not met. We shall see later (Example 2, § 65) that in this case $\phi(\alpha) = 0$, so that the function is continuous for $\alpha = \pm 1$.

61. Integration under the integral sign. The possibility of differentiating under the integral sign leads conversely to the possibility of integration. Let

$$\phi(\alpha) = \int_a^b f(x, \alpha)dx, \tag{1}$$

where a and b are constants. Multiply by $d\alpha$ and integrate with respect to α between α_0 and α. Then

$$\int_{\alpha_0}^{\alpha} \phi(\alpha)d\alpha = \int_{\alpha_0}^{\alpha} d\alpha \int_a^b f(x, \alpha)dx, \tag{2}$$

where the integrations on the right are to be carried out from right to left.

On the other hand, consider

$$\Phi(\alpha) = \int_a^b dx \int_{\alpha_0}^{\alpha} f(x, \alpha)d\alpha. \tag{3}$$

We wish to show that (3) and (2) are the same. We differentiate (3) with respect to α. By the previous section the differentiation on the right may be carried out under the integral sign, and by (1), § 60,

$$\frac{\partial}{\partial\alpha}\int_{\alpha_0}^{\alpha} f(x, \alpha)d\alpha = f(x, \alpha).$$

Hence, from (3),

$$\Phi'(\alpha) = \int_a^b f(x, \alpha)dx = \phi(\alpha). \tag{4}$$

Then

$$\int_{\alpha_0}^{\alpha} \Phi'(\alpha)d\alpha = \int_{\alpha_0}^{\alpha} \phi(\alpha)d\alpha,$$

or

$$\Phi(\alpha) = \int_{\alpha_0}^{\alpha} \phi(\alpha)d\alpha, \tag{5}$$

since, from (3), $\Phi(\alpha_0) = 0$. Hence, by replacing the two members of equation (5) by their respective values as given in (3) and (2), we have

$$\int_a^b dx \int_{\alpha_0}^{\alpha} f(x, \alpha)d\alpha = \int_{\alpha_0}^{\alpha} d\alpha \int_a^b f(x, \alpha)dx, \tag{6}$$

as was to be proved.

This may be considered either as proving the method of integrating under the integral sign or as showing the possibility of interchanging the order of repeated integration.

For example, consider

$$\int_0^1 x^\alpha\, dx = \frac{1}{\alpha+1}. \quad (\alpha + 1 > 0)$$

Multiply by $d\alpha$ and integrate between a and b. Then

$$\int_a^b d\alpha \int_0^1 x^\alpha\, dx = \int_a^b \frac{d\alpha}{\alpha+1} = \log\frac{b+1}{a+1}.$$

But, by (6), the left-hand side of this equation is equal to

$$\int_0^1 dx \int_a^b x^\alpha\, d\alpha = \int_0^1 \frac{x^b - x^a}{\log x}\, dx,$$

and therefore
$$\int_0^1 \frac{x^b - x^a}{\log x}\, dx = \log\frac{b+1}{a+1}.$$

62. Infinite limit. It is possible to consider definite integrals with the upper limit infinity if we place by definition

$$\int_a^\infty f(x)dx = \operatorname*{Lim}_{b\to\infty} \int_a^b f(x)dx. \qquad (1)$$

The proof of the existence of the integral now breaks down, and there is no guaranty that the limit in (1) actually exists. When it does, we say that the integral (1) converges. It is important to know something of the conditions under which this takes place.

If it is possible to evaluate the definite integral by the formula

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F(x)$ is an elementary function which may be explicitly found, the convergence of the integral (1) may be determined by examining the behavior of $F(b)$ as b increases indefinitely. For example, consider

$$\int_1^b \frac{dx}{x^k}.$$

If $k = 1$, this integral is
$$\int_1^b \frac{dx}{x} = \log b;$$

if $k \neq 1$, it is
$$\int_1^b \frac{dx}{x^k} = \frac{b^{1-k}}{1-k} - \frac{1}{1-k}.$$

As $b \to \infty$, $\log b \to \infty$ and $b^{1-k} \to \infty$ if $k < 1$, while $b^{1-k} \to 0$ if $k > 1$. Hence we have the following theorem:

I. The integral $\int_1^\infty \frac{dx}{x^k}$ *converges if* $k > 1$ *and becomes infinite if* $k \leqq 1$.

We may now prove the following theorem:

II. When the integral $\int_a^\infty f(x)dx$ *can be written in the form*

$$\int_a^\infty \frac{\phi(x)}{x^k}\,dx,$$

then (1) if $\phi(x)$ *is less in absolute value than a finite number* M *for sufficiently large values of* x, *and* $k > 1$, *the integral converges; (2) if* $\phi(x)$ *is greater in absolute value than a positive number* m *for sufficiently large values of* x, *and* $k \leqq 1$, *the integral does not converge.*

To prove this let us write

$$\int_a^b f(x)dx = \int_a^l f(x)dx + \int_l^\infty f(x)dx;$$

then if $\int_a^b f(x)dx$ converges, it is necessary and sufficient that $\left|\int_l^\infty f(x)dx\right| < \epsilon$ for sufficiently large l. Consider, then, the first part of the theorem given above. We have

$$\left|\int_l^\infty f(x)dx\right| = \left|\int_l^\infty \frac{\phi(x)}{x^k}\,dx\right| < M\int_l^\infty \frac{dx}{x^k} = M\,\frac{l^{k-1}}{k-1},$$

and by taking l sufficiently large this can be made less than any quantity ϵ.

On the other hand, under the second part of the theorem

$$\left|\int_l^\infty f(x)dx\right| = \left|\int_l^\infty \frac{\phi(x)}{x^k}\,dx\right| > m\int_l^\infty \frac{dx}{x^k},$$

where the last integral increases beyond any limit.

As examples consider the following, all of which are integrals which cannot be evaluated by elementary means.

Example 1. $\int_a^\infty \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$.

If we take
$$\phi(x) = \frac{1}{\sqrt{\left(\frac{1}{x^2}-1\right)\left(\frac{1}{x^2}-k^2\right)}},$$

it is evident that as $x \rightarrow \infty$, $\phi(x) \rightarrow \frac{1}{k}$; and hence $\phi(x)$ is finite as $x \rightarrow \infty$, and, for sufficiently large x, $\phi(x) < \frac{2}{k}$ or some other quantity chosen at pleasure. The integral is $\int_a^\infty \frac{\phi(x)}{x^2}\,dx$, and consequently converges.

Example 2. $\int_0^\infty e^{-x^2}\,dx.$

Take $\phi(x) = x^2e^{-x^2}$. Then $\text{Lim}\, x^2e^{-x^2} = 0$ (§ 10); and, for sufficiently large values of x, $\phi(x) < 1$. The integral is $\int_0^\infty \frac{\phi(x)}{x^2}\,dx$, which converges.

Example 3. $\int_0^\infty \frac{\sin x}{x}\,dx.$

No conclusion can be drawn from the theorem for this integral. If we place $\phi(x) = \sin x$, the absolute value of $\phi(x)$ is always less than 1. But $k = 1$, and therefore the first part of the theorem does not apply. On the other hand, there is no positive quantity $m < \phi(x)$ for all large values of x, and therefore the second part of the theorem does not apply.

The third example shows that the theorem we have given is not sufficient to determine the convergence of all integrals, but its range of applicability is large. The convergence of the integral in Example 3 may be established in another way. If we graph the function $y = \frac{\sin x}{x}$, the graph consists of portions alternately above and below the axis of x, and it is evident that the integral may be written

$$\int_0^\infty \frac{\sin x}{x}\,dx = u_1 - u_2 + u_3 - u_4 + \cdots, \tag{2}$$

where u_k is the absolute value of the integral

$$\int_{(k-1)\pi}^{k\pi} \frac{\sin x}{x}\,dx.$$

Now as $\frac{1}{x}$ is constantly decreasing without limit, it is evident that

$$u_{k+1} < u_k \quad \text{and} \quad \underset{k\to\infty}{\text{Lim}}\, u_k = 0.$$

By § 29 the series (2) converges.

63. Differentiation and integration of an integral with an infinite limit. The question naturally arises, When is

$$\phi(\alpha) = \int_a^\infty f(x, \alpha)dx \tag{1}$$

a continuous function of α and when may it be differentiated under the integral sign? We shall not endeavor to give a complete answer to this question, but will give certain rules which are sometimes applicable. We shall assume that $f(x, \alpha)$ is a continuous function of x and α for any value of x between a and ∞ and for any value of α in an interval (α_0, α_1).

Let us write (1) in the form

$$\int_a^\infty f(x, \alpha)dx = \int_a^l f(x, \alpha)dx + \int_l^\infty f(x, \alpha)dx, \tag{2}$$

where l is a large but finite quantity. If, now, ϵ being any assigned positive quantity, it is possible to choose l so that

$$\left| \int_l^\infty f(x, \alpha)dx \right| < \epsilon \tag{3}$$

for all values of α in the interval (α_0, α_1), then (1) is said to converge uniformly in the interval (α_0, α_1). We may then prove that (1) is a continuous function of α. For we have

$$\Delta\phi = \int_a^l [f(x, \alpha + \Delta\alpha) - f(x, \alpha)]dx + \int_l^\infty f(x, \alpha + \Delta\alpha)dx \\ - \int_l^\infty f(x, \alpha)dx.$$

By hypothesis we may so choose l that each of the last integrals is less absolutely than $\frac{\epsilon}{3}$. We may then choose $\Delta\alpha$ so that

$$| f(x, \alpha + \Delta\alpha) - f(x, \alpha) | < \frac{\epsilon}{3(l - a)},$$

since, by hypothesis, $f(x, \alpha)$ is a continuous function.

Then $$|\Delta\phi| < \epsilon,$$

and hence (1) is a continuous function of α.

To differentiate (1) we shall also assume that $f_\alpha(x, \alpha)$ is a continuous function of x and α for all values of x in the interval (a, ∞) and for all values of α in the interval (a_0, α_1), and that the integral $\int_a^\infty f_\alpha(x, \alpha)dx$ converges uniformly. We have, now,

$$\frac{\Delta\phi}{\Delta\alpha} = \int_a^l \frac{f(x, \alpha + \Delta\alpha) - f(x, \alpha)}{\Delta\alpha} dx + \int_l^\infty \frac{f(x, \alpha + \Delta\alpha) - f(x, \alpha)}{\Delta\alpha} dx$$

$$= \int_a^l f_\alpha(x, \alpha + \theta\, \Delta\alpha)dx + \int_l^\infty f_\alpha(x, \alpha + \theta\, \Delta\alpha)dx. \quad (0 < \theta < 1)$$

Since, by hypothesis, $\int_a^\infty f_\alpha(x, \alpha)dx$ converges uniformly, we may choose l so that the last integral is less absolutely than $\frac{\epsilon}{2}$. Since f_α is continuous by hypothesis, we may choose $\Delta\alpha$ so that

$$f_\alpha(x, \alpha + \theta\,\Delta\alpha) = f_\alpha(x, \alpha) + \eta_1,$$

where $|\eta_1| < \frac{\epsilon}{2(l - \alpha)}$.

Hence
$$\frac{\Delta\phi}{\Delta\alpha} = \int_a^l f_\alpha(x, \alpha)dx + \eta,$$

where $|\eta| < \epsilon$. Since, by hypothesis, the integral approaches a limit as $l \to \infty$, we have

$$\frac{d\phi}{d\alpha} = \int_a^\infty f_\alpha(x, \alpha)dx. \tag{4}$$

In order to apply the theorem it is necessary to be able to determine whether the integrals involved converge absolutely or not. This may often be done by finding a positive function $\phi(x)$ such that

$$\phi(x) \geqq f(x, \alpha) \tag{5}$$

for all values of x in the interval (l, ∞) and for all values of α in the interval (α_0, α_1). Then if

$$\int_l^\infty \phi(x)dx$$

converges, the integral $\int_l^\infty f(x, \alpha)dx$ converges uniformly. This is obvious from the relation

$$\left|\int_l^\infty f(x, \alpha)dx\right| < \int_l^\infty \phi(x)dx < \epsilon,$$

where ϵ can be as small as we please by taking l sufficiently great.

As an example, consider

$$\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\,dx. \quad (\alpha > 0) \tag{6}$$

Here $\quad f(x, \alpha) = \frac{e^{-\alpha x}\sin x}{x}, \qquad f_\alpha(x, \alpha) = -\,e^{-\alpha x}\sin x,$

and f and f_α satisfy the conditions of being continuous functions of x and α if $\alpha > 0$ and $0 < x < l$, where l is any positive number no matter how large. We may write

$$\frac{e^{-\alpha x}\sin x}{x} = \frac{xe^{-\alpha x}\sin x}{x^2}.$$

Now $\operatorname*{Lim}_{x \to \infty} xe^{-\alpha x} = 0$. Therefore for sufficiently large values of x, $xe^{-\alpha x} < 1$; and since $|\sin x| \overline{\gtrless} 1$, we have

$$\left|\frac{xe^{-\alpha x}\sin x}{x^2}\right| < \frac{1}{x^2}.$$

Then $\frac{1}{x^2}$ is the function $\phi(x)$ of (5) and condition (5) is met, and therefore (6) defines a continuous function of α.

Again, by writing $\quad e^{-\alpha x}\sin x = \frac{x^2 e^{-\alpha x}\sin x}{x^2}$,

we see that for sufficiently large values of x, $|e^{-\alpha x}\sin x| < \frac{1}{x^2}$, and hence $f_\alpha(x, \alpha)$ meets the required conditions. Therefore formula (4) holds for the integral (6).

An integral with an infinite limit may be integrated under the integral sign if the condition of absolute convergence is met by the integrals involved. This is proved as in § 61.

64. Infinite integrand. In our discussion of the definite integral thus far it has been necessary that $f(x)$ should remain finite in the interval of integration. It is the purpose of this section to examine certain cases in which $f(x)$ becomes infinite at one or more points. It will be sufficient to examine the case in which $f(x)$ becomes infinite at the upper limit $x = b$. For if $f(x)$ becomes infinite when $x = a$, that limit may be made the upper limit by changing the sign of the integral (§ 56); if $f(x)$ becomes infinite at any intermediate point c, we may use (4), § 56, and examine each integral separately.

If, then, $f(b) \to \infty$, we define the integral by the formula

$$\int_a^b f(x)dx = \operatorname*{Lim}_{\epsilon \to 0} \int_a^{b-\epsilon} f(x)dx. \tag{1}$$

When the limit exists, the integral is said to *converge*.

When the integral may be evaluated by the formula

$$\int_a^b f(x)dx = F(b) - F(a),$$

the convergence may be examined by considering the behavior of $F(b - \epsilon)$ as $\epsilon \to \infty$. For example,

$$\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \operatorname*{Lim}_{\epsilon \to 0} \int_0^{a-\epsilon} \frac{dx}{\sqrt{a^2 - x^2}} = \operatorname*{Lim}_{\epsilon \to 0} \sin^{-1}\frac{a - \epsilon}{a} = \frac{\pi}{2}.$$

Again, consider $\displaystyle\int_a^{b-\epsilon}\frac{dx}{(b-x)^k}.$

If $k=1$, this becomes

$$\int_a^{b-\epsilon}\frac{dx}{b-x}=-\log\epsilon+\log(b-a);$$

if $k\neq 1$,
$$\int_a^{b-\epsilon}\frac{dx}{(b-x)^k}=\frac{(b-a)^{1-k}-\epsilon^{1-k}}{1-k}.$$

From these results the following theorem is at once evident:

I. The integral $\displaystyle\int_a^b\frac{dx}{(b-x)^k}$ *converges if* $k<1$ *and diverges if* $k\geqq 1$.

From this we may deduce the following theorem:

II. When the integral $\displaystyle\int_a^b f(x)dx$ *can be written in the form*

$$\int_a^b\frac{\phi(x)}{(b-x)^k}dx,$$

then (1) if, for values of x *sufficiently near* b, $\phi(x)$ *is less in absolute value than a number* M, *and* $k<1$, *the integral converges; (2) if, for values of* x *sufficiently near* b, $\phi(x)$ *is greater in absolute value than a positive number* m, *and* $k\geqq 1$, *the integral diverges.*

For example, the integral $\displaystyle\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ may be written $\displaystyle\int_0^1\frac{\phi(x)}{\sqrt{1-x}}dx$, where $\phi(x)=\dfrac{1}{\sqrt{(1+x)(1-k^2x^2)}}$, and therefore converges. The integral $\displaystyle\int_0^a\frac{dx}{(x-a)\sqrt{(1-x^2)(1-k^2x^2)}}$ evidently diverges.

By repeating in essence the proof in the previous section we may show that an integral with infinite limit $b=\infty$ may be differentiated or integrated under the integral sign provided the functions $f(x,\alpha)$ and $f_\alpha(x,\alpha)$ are continuous and that the conditions for uniform convergence are met; that is,

$$\left|\int_l^b f(x,\alpha)dx\right|<\eta \quad\text{and}\quad \left|\int_l^b\frac{\partial f}{\partial\alpha}dx\right|<\eta$$

for all values of α between α_0 and α_1 when l is chosen sufficiently near b, and η is any assumed positive quantity.

65. Certain definite integrals. We shall discuss in this section certain definite integrals of importance by means of certain special devices.

Example 1. $\int_0^\infty e^{-x^2}\,dx$. This integral has been shown to converge.

Let $$I = \int_0^b e^{-x^2}\,dx = \int_0^b e^{-y^2}\,dy. \tag{1}$$

We may use either y or x in writing the integral, since the form of the function and the limits only are essential. Then

$$I^2 = \int_0^b e^{-x^2}\,dx \int_0^b e^{-y^2}\,dy = \int_0^b \int_0^b e^{-(x^2+y^2)}\,dx\,dy, \tag{2}$$

where the double integral is taken over the square $OACB$ (Fig. 58).

Since all the terms of the sum in (2) are positive, the value of I^2 is greater than would be the same sum taken over the quadrant of a circle of radius $OB = b$ and less than the sum taken over a quadrant of radius

$$OC = b\sqrt{2}.$$

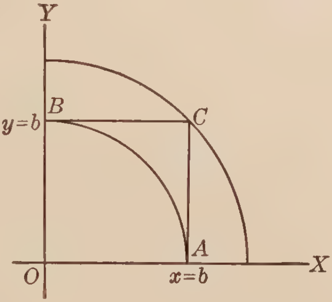

FIG. 58

In summing over the quadrants we may use polar coördinates and have

$$\int_0^{\frac{\pi}{2}}\int_0^b e^{-r^2} r\,d\theta\,dr < I^2 < \int_0^{\frac{\pi}{2}}\int_0^{b\sqrt{2}} e^{-r^2} r\,d\theta\,dr;$$

whence $$\frac{\pi}{4}(1 - e^{-b^2}) < I^2 < \frac{\pi}{4}(1 - e^{-2b^2}). \tag{3}$$

Now let $b \to \infty$. By definition I approaches $\int_0^\infty e^{-x^2}\,dx$, and the first and last members of the inequality (3) approach $\frac{\pi}{4}$. Therefore

$$\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}. \tag{4}$$

Example 2. $\int_0^{\frac{\pi}{2}} \log \sin x\,dx$. The function $\log \sin x$ becomes infinite when $x = 0$; but we may write $\log \sin x = \frac{\phi(x)}{\sqrt{x}}$, where $\phi(x) = x^{\frac{1}{2}} \log \sin x$, and show by § 10 that $\lim\limits_{x \to 0} \phi(x) = 0$. Therefore, by § 64, the integral converges.

Let $$u = \int_0^{\frac{\pi}{2}} \log \sin x\,dx = \int_0^{\frac{\pi}{2}} \log \cos y\,dy. \quad \left(y = \frac{\pi}{2} - x\right) \tag{5}$$

We may replace y by x in the last integral of (5). Then

$$2u = \int_0^{\frac{\pi}{2}} [\log \sin x + \log \cos x]\,dx = \int_0^{\frac{\pi}{2}} \log \sin 2x\,dx - \int_0^{\frac{\pi}{2}} \log 2\,dx. \tag{6}$$

Integrating the last integral of (6) and placing $2x = z$ in the next to the last integral, we have

$$2u + \frac{\pi}{2}\log 2 = \frac{1}{2}\int_0^{\pi} \log \sin z \, dz$$
$$= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \log \sin z \, dz + \tfrac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \log \sin z \, dz. \qquad (7)$$

In the next to the last integral of (7) we place $z = x$ and in the last integral we place $z = \frac{\pi}{2} - y$ and use (5). Then

$$2u + \frac{\pi}{2}\log 2 = \frac{1}{2}u + \frac{1}{2}u;$$

whence
$$u = -\frac{\pi}{2}\log 2. \qquad (8)$$

This result enables us to complete the discussion of (11), § 60. For if we place $\alpha = 1$ we have

$$\phi(1) = \int_0^{\pi} \log 2(1 - \cos x)dx = \int_0^{\pi} \log\left(4 \sin^2 \frac{x}{2}\right)dx$$
$$= \log 4 \int_0^{\pi} dx + 2\int_0^{\pi} \log \sin \frac{x}{2}\, dx$$
$$= \pi \log 4 + 4\int_0^{\frac{\pi}{2}} \log \sin y \, dy \qquad \left(y = \frac{x}{2}\right)$$
$$= \pi \log 4 + 4\left(-\frac{\pi}{2}\log 2\right) = 0.$$

Example 3. $\int_0^{\infty} \frac{e^{-\alpha x}\sin x}{x}\, dx. \quad (\alpha > 0)$

We have seen that this integral defines a continuous function of α and that it may be differentiated under the integral sign. Then

$$\phi(\alpha) = \int_0^{\infty} \frac{e^{-\alpha x}\sin x}{x}\, dx.$$
$$\phi'(\alpha) = -\int_0^{\infty} e^{-\alpha x} \sin x \, dx$$
$$= -\frac{1}{1 + \alpha^2}.$$

Therefore
$$\phi = -\tan^{-1}\alpha + C.$$

Now as $\alpha \rightarrow \infty$, $\phi(\alpha) \rightarrow 0$, and therefore $C = \frac{\pi}{2}$;

whence
$$\phi(\alpha) = \cot^{-1}\alpha = \tan^{-1}\frac{1}{\alpha}.$$

Example 4. $\int_0^\infty \frac{\sin x}{x}\,dx.$

This integral arises from Example 3 when $\alpha = 0$. The evaluation of Example 3 depends, however, upon the assumption that $\alpha \neq 0$. We cannot place $\alpha = 0$ in Example 3, therefore, without first showing that $\phi(\alpha)$ is a continuous function of α. This we may do as follows. We write

$$\phi(\alpha) = \int_0^\infty \frac{e^{-\alpha x} \sin x}{x}\,dx.$$

$$\phi(0) = \int_0^\infty \frac{\sin x}{x}\,dx.$$

We have already shown that $\phi(\alpha)$ and $\phi(0)$ converge. In fact, we have, as in Example 3, § 62,

$$\phi(\alpha) = u_1(\alpha) - u_2(\alpha) + u_3(\alpha) - \cdots \pm u_n(\alpha) \mp \cdots,$$
$$\phi(0) = u_1(0) - u_2(0) + u_3(0) - \cdots \pm u_n(0) \mp \cdots,$$

where the limits of integration for $u_k(\alpha)$ and $u_k(0)$ are the same. Let ϵ be any assigned positive number. We may take a finite number of terms n so that the remainder after n times in each of these series is less than $\frac{\epsilon}{3}\cdot$ Then

$$\phi(\alpha) - \phi(0) = [u_1(\alpha) - u_1(0)] - [u_2(\alpha) - u_2(0)] + \cdots$$
$$\pm [u_n(\alpha) - u_n(0)] + \eta,$$

where $|\eta| < \frac{2\epsilon}{3}\cdot$ Now each of the functions $u_k(\alpha)$ is a continuous function when $\alpha = 0$. Hence we may take α so small that each of the terms $|u_k(\alpha) - u_k(0)| < \frac{\epsilon}{2n}\cdot$ Then

$$|\phi(\alpha) - \phi(0)| < \epsilon,$$

which shows that $\phi(\alpha)$ is continuous at $\alpha = 0$.

Therefore $$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{\alpha \to 0}\left(\tan^{-1}\frac{1}{\alpha}\right) = \frac{\pi}{2}\cdot$$

Example 5. $\int_0^\infty e^{-\alpha^2 x^2}\,dx.$

Place $\alpha x = y$. Then, by Example 1,

$$\int_0^\infty e^{-\alpha^2 x^2}\,dx = \frac{1}{\alpha}\int_0^\infty e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2\alpha}\cdot$$

Example 6. $\int_0^\infty \sin x^2\,dx$ and $\int_0^\infty \cos x^2\,dx.$

To obtain these integrals we shall use certain properties of complex numbers, thus anticipating Chapter XV. This chapter may be consulted in advance or the reading of this example may be postponed.

By (5), § 26, $$e^{-ix^2} = \cos x^2 - i \sin x^2.$$

We shall therefore study

$$\int_0^\infty e^{-ix^2}\,dx = \int_0^\infty \cos x^2\,dx - i\int_0^\infty \sin x^2\,dx,$$

which can be shown to converge by showing, as in Example 3, § 62, that each of the two integrals on the right converges.

By Example 5, $$\int_0^\infty e^{-ix^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{i}}.$$

Now $\frac{1}{\sqrt{i}} = \frac{\sqrt{2}}{1+i}$, as may be verified by squaring both sides of the equation, and $\frac{\sqrt{2}}{1+i} = \frac{1-i}{\sqrt{2}}.$

Hence $$\int_0^\infty \cos x^2\,dx - i\int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,(1-i);$$

and therefore, by equating real and imaginary parts,

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

66. Multiple integrals. Let a region R (Fig. 59) be given in the xy-plane, and let it be divided into rectangles by lines

$$x = x_i \quad (i = 0, 1, 2, \cdots, n)$$

and $y = y_k$. $(k = 0, 1, 2, \cdots, m)$

Most of these rectangles lie inside the region R, but at the boundary some will project out of the region. Consider any rectangle, with dimensions $x_{i+1} - x_i$ and $y_{k+1} - y_k$, which lies either wholly or partly in R and let (ξ_i, η_k) be any point in this rectangle and also in R if the rectangle extends outside of R.

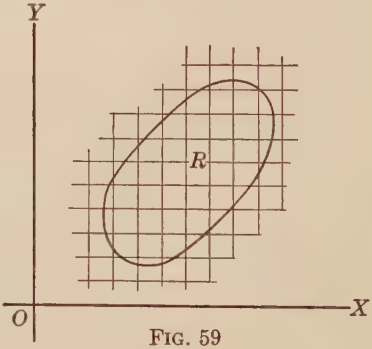

FIG. 59

Let $f(x, y)$ be a function which is continuous in the region R. Then it may be shown that the sum

$$\sum_{i=0}^{i=n}\sum_{k=0}^{k=m} f(\xi_i, \eta_k)(x_{i+1} - x_i)(y_{k+1} - y_k)$$

approaches a limit as n and m increase indefinitely and each of the differences $x_{i+1} - x_i$ and $y_{k+1} - y_k$ approaches zero. We shall omit the proof, which proceeds on lines analogous to those used in § 55. This limit is expressed by

$$\iint\limits_{(R)} f(x, y)dx\,dy, \tag{1}$$

and the function $f(x, y)$ is said to be summed over the region R. It is assumed that the student is familiar, from his study of elementary calculus, with illustrations of the use of double integrals.

In the foregoing discussion the region R may be any shape whatever, with any number of distinct boundary curves. In case the region is of the shape sketched in Fig. 60, where any line parallel to OY meets the boundary curve in two points for which the values of y are $y_1 = f_1(x)$ and $y_2 = f_2(x)$, and the extreme values of x are $x = a$ and $x = b$, the integral may be evaluated by the formula

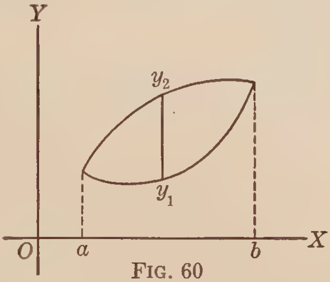

FIG. 60

$$\iint\limits_{(R)} f(x, y)dx\,dy = \int_a^b dx \int_{y_1}^{y_2} f(x, y)dy; \tag{2}$$

or in case a line parallel to OX meets the boundary curve in two points $x = x_1$, $x = x_2$, and the extreme values of y are $y = c$, $y = d$, then also

$$\iint\limits_{(R)} f(x, y)dx\,dy = \int_c^d dy \int_{x_1}^{x_2} f(x, y)dx. \tag{3}$$

In case the area over which the integration takes place is a rectangle bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$, formulas (2) and (3) yield the result

$$\int_a^b dx \int_c^d f(x, y)dy = \int_c^d dy \int_a^b f(x, y)dx, \tag{4}$$

which embodies the principle of interchange of the order of integration.

Formula (4) is always valid if the limits a, b, c, d are finite and $f(x, y)$ is continuous in the rectangle. Without formal proof this statement is geometrically plausible if we consider the integrals in (4) as defining a volume.

If one or more of the limits in (4) becomes infinite or if $f(x, y)$ has discontinuities, the formula is not necessarily true. The student will often meet examples, however, in which (4) is applied to such cases. This can usually be justified by the theorem* that if $f(x, y)$ does not change its sign, formula (4) holds, provided the integral $\int_c^d f(x, y)dy$ is a continuous function of x and the integral $\int_a^b f(x, y)dx$ is a continuous function of y, except perhaps for isolated points. To determine when the simple integrals satisfy the conditions demanded, the tests of § 62 and § 64 are usually sufficient.

The integration over a more complicated region R may be carried out by separating that region into smaller regions of the simpler type just considered, but in this text we shall be more concerned with the properties of a definite integral than with its evaluation, which is a subject for the elementary calculus.

We may write the integral (1) in the form

$$\iint\limits_{(R)} f(x, y)dA, \tag{5}$$

where dA is the element of area $dx\,dy$. In case the coördinates (x, y) are replaced by curvilinear coördinates (u, v), then, as shown in § 53, dA is to be replaced by

$$\pm J\left(\frac{x, y}{u, v}\right)du\,dv, \tag{6}$$

the sign being so chosen as to make the area positive. On the other hand, in $f(x, y)$ we have simply to make the substitution (5). That the integral obtained in this way is exactly the same as the original we shall leave as sufficiently plausible, without making the careful analysis necessary for rigorously proving this.

Again, let a region of space R be divided into rectangular parallelepipeds by planes parallel to the coördinate axes in a manner analogous to the division of the plane. Let the vertex of one such parallelepiped, which lies either entirely or partly in R, be (x_i, y_j, z_k), let its edges be $x_{i+1} - x_i$, $y_{j+1} - y_j$, $z_{k+1} - z_k$, and let (ξ_i, η_j, ζ_k) be a point in its interior and in R. Then, if $f(x, y, z)$ is a function continuous in R, the sum

$$\sum_i \sum_j \sum_k f(\xi_i, \eta_j, \zeta_k)(x_{i+1} - x_i)(y_{j+1} - y_j)(z_{k+1} - z_k)$$

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* De la Valée Poussin, Cours d'Analyse, 4th ed., Vol. II, p. 23.

approaches a limit as the number of parallelepipeds increases indefinitely and the edges of each approach zero. This limit is the triple integral

$$\iiint_{(R)} f(x, y, z)dx\,dy\,dz. \tag{7}$$

The region R may be of any shape. If it is such that a line parallel to OZ enters the region through a surface $z_1 = f_1(x, y)$ and leaves it through the surface $z_2 = f_2(x, y)$, then (7) may be written

$$\iint_{(S)} dx\,dy \int_{z_1}^{z_2} f(x, y, z)dz, \tag{8}$$

where the region S, over which the double integral is taken, is the projection of R on XOY.

The triple integral (7) may be written

$$\iiint_{(R)} f(x, y, z)dV, \tag{9}$$

where dV is the element of volume $dx\,dy\,dz$. If it is desired to use curvilinear coördinates (u, v, w), as shown in § 53,

$$dV = \pm J\left(\frac{x, y, z}{u, v, w}\right)du\,dv\,dw. \tag{10}$$

We have already seen that in cylindrical coördinates we have

$$dV = r\,d\theta\,dr\,dz, \tag{11}$$

and in polar coördinates,

$$dV = r^2 \sin\phi\,d\theta\,d\phi\,dr. \tag{12}$$

EXERCISES

1. If $f(x)$ is an odd function, that is, if $f(-x) = -f(x)$, prove that

$$\int_{-a}^{a} f(x)dx = 0.$$

2. If $f(x)$ is an even function, that is, if $f(-x) = f(x)$, prove that

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$

3. If $f(a - x) = f(x)$, prove that

$$\int_0^a f(x)dx = 2\int_0^{\frac{1}{2}a} f(x)dx.$$

4. Show that $\int_0^{2k\pi} f(\sin x)dx = k\int_0^{2\pi} f(\sin x)dx.$

5. If $f(x)$ has a period a, that is, if $f(x+a)=f(x)$, prove that

$$\int_0^{ka} f(x)dx = k\int_0^a f(x)dx. \quad (k = \text{any integer})$$

6. If $a < b$, and $f_1(x) < f_2(x) < f_3(x)$ for any x in the interval (a, b), prove that

$$\int_a^b f_1(x)dx < \int_a^b f_2(x)dx < \int_a^b f_3(x)dx.$$

7. If m and M are the smallest and the largest values of $f(x)$ in the interval (a, b), and $\phi(x) > 0$ in the interval, prove that

$$m\int_a^b \phi(x)dx < \int_a^b f(x)\phi(x)dx < M\int_a^b \phi(x)dx$$

and therefore

$$\int_a^b f(x)\phi(x)dx = f(\xi)\int_a^b \phi(x)dx. \quad (a < \xi < b)$$

8. Evaluate
$$\int_0^3 (1+x^2)^{\frac{3}{2}}dx$$
by Simpson's rule, taking $n = 3$.

9. Evaluate
$$\int_1^3 \frac{dx}{(1+x^2)^2}$$
by Simpson's rule, taking $n = 2$.

10. Evaluate
$$\int_0^{\frac{\pi}{3}} \log_{10} \cos x \, dx$$
by Simpson's rule, taking $n = 2$.

11. Examine the integral $\int_0^1 \frac{dx}{\sqrt{\alpha^2 + x^2}}$ for continuity when $\alpha = 0$.

12. Examine the integral $\int_0^1 \frac{\alpha\, dx}{\sqrt{\alpha^2 + x^2}}$ in the neighborhood of $\alpha = 0$.

Find the derivatives, with respect to α, of the following integrals without first integrating, and check by first integrating and then differentiating:

13. $\int_0^{\alpha x} \cos(x+\alpha)dx.$

14. $\int_0^x \sin^{-1}\frac{x}{\alpha}\, dx.$

15. $\int_0^{\sqrt{\alpha}} x\, dx.$

16. $\int_{\alpha^2}^{\alpha^3} (x^2 + \alpha^2)dx.$

By differentiating with respect to α, find the values of the following integrals:

17. $\int_0^{\pi} \log(1 + \alpha\cos x)dx.$

18. $\int_0^1 \frac{x^\alpha - 1}{\log x}\, dx.$

19. By successive differentiations of $\int_0^1 x^n\,dx = \dfrac{1}{n+1}$ obtain

$$\int_0^1 x^n(\log x)^m\,dx = (-1)^m\frac{m!}{(n+1)^{m+1}}.$$

20. From

$$\int_0^\pi \frac{dx}{\alpha-\cos x} = \frac{\pi}{\sqrt{\alpha^2-1}} \qquad (\alpha > 1)$$

find

$$\int_0^\pi \log\frac{b-\cos x}{a-\cos x} = \pi\log\frac{b+\sqrt{b^2-1}}{a+\sqrt{a^2-1}}.$$

Test the convergence of the following integrals:

21. $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}.$

22. $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$

23. $\displaystyle\int_a^\infty \frac{x^4\,dx}{(x^2+a^2)^{\frac{5}{2}}}.$

24. $\displaystyle\int_0^\infty e^{-a^2x^2}\cos bx\,dx.$

25. $\displaystyle\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx.$

26. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^3-1}}.$

27. By methods analogous to those used in Example 3, § 62, of the text, prove the convergence of

$$\int_0^\infty \sin x^2\,dx.$$

28. Prove the convergence of

$$\int_0^\infty \frac{e^{-ax}\sin mx}{x}\,dx.$$

Prove that the following integrals satisfy the conditions for differentiability with respect to α under the integral sign:

29. $\displaystyle\int_0^\infty e^{-\alpha x}\,dx.$

30. $\displaystyle\int_0^\infty e^{-\alpha x^2}\,dx.$

31. $\displaystyle\int_0^\infty e^{-bx^2}\cos\alpha x\,dx.$

32. $\displaystyle\int_0^\infty \frac{dx}{x^2+\alpha}.$

33. From $\displaystyle\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}$ obtain by differentiation

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}.$$

34. From $\displaystyle\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$ obtain by differentiation

$$\int_0^\infty x^{2n}e^{-\alpha x^2}\,dx = \frac{\sqrt{\pi}}{2}\,\frac{1\cdot 3\cdots(2n-1)}{2^n\,\alpha^{n+\frac{1}{2}}}.$$

35. From $\displaystyle\int_0^\infty \frac{dx}{x^2+\alpha}\,dx = \frac{\pi}{2\sqrt{\alpha}}$ obtain by differentiation

$$\int_0^\infty \frac{dx}{(x^2+\alpha)^{n+1}} = \frac{\pi}{2}\,\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n\,\alpha^{n+\frac{1}{2}}}.$$

36. From $\int_0^\infty e^{-\alpha x}\cos mx\,dx = \frac{\alpha}{\alpha^2+m^2}$ obtain by integration

$$\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x\sec mx}\,dx = \frac{1}{2}\log\frac{\beta^2+m^2}{\alpha^2+m^2}.$$

37. From $\int_0^\infty e^{-\alpha x}\sin mx\,dx = \frac{m}{\alpha^2+m^2}$ find by integration

$$\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x\csc mx}\,dx = \tan^{-1}\frac{\beta}{m} - \tan^{-1}\frac{\alpha}{m}.$$

38. From $\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}$ obtain by integration

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \log\frac{b}{a}.$$

39. From $\int_0^\infty e^{-\alpha^2x^2}\,dx = \frac{\pi}{2\sqrt{\alpha}}$ obtain by integration

$$\int_0^\infty \left(\frac{e^{-a^2x^2}-e^{-b^2x^2}}{x^2}\right)dx = (b-a)\sqrt{\pi}.$$

Investigate the convergence of the following integrals:

40. $\int_0^1 (\log x)^n\,dx.$

41. $\int_0^1 \frac{\log x}{1-x^2}\,dx.$

42. $\int_0^1 \frac{x\,dx}{(1-x^4)^{\frac{1}{3}}}.$

43. $\int_0^\infty \frac{dx}{x^{\frac{2}{3}}(1+x)}.$

44. $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}.$

45. $\int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx.$

Evaluate the following integrals. The results of the chapter and any elementary integrals may be used; and in any exercise the result of a previous exercise may be used. In some cases a change of variable is the only step necessary.

46. $$\int_0^\infty \frac{\sin mx}{x}\,dx = \frac{\pi}{2} \quad \text{if} \quad m>0,$$
$$= 0 \quad \text{if} \quad m=0,$$
$$= -\frac{\pi}{2} \quad \text{if} \quad m<0.$$

47. $$\int_0^\pi x\log\sin x\,dx = -\frac{\pi^2}{2}\log 2.$$

48. $$\int_0^\infty e^{-\alpha^2x^2}\,dx = \frac{1}{2\,\alpha}\sqrt{\pi}.$$

49. $$\int_0^1 \frac{dx}{\sqrt{\log\frac{1}{x}}} = \sqrt{\pi}.$$

50. $$\int_0^\infty \frac{\sin x \cos mx}{x}\,dx = 0 \quad \text{if} \quad m < -1 \quad \text{or} \quad m > 1,$$
$$= \frac{\pi}{4} \quad \text{if} \quad m = -1 \quad \text{or} \quad m = 1,$$
$$= \frac{\pi}{2} \quad \text{if} \quad -1 < m < 1.$$

51. $$\int_0^\infty e^{-x^2 - \frac{a^2}{x^2}}\,dx = \frac{e^{-2a}\sqrt{\pi}}{2}.$$

HINT. Representing the integral by u, first show that $\frac{du}{da} = -2u$.

52. $$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}.$$

53. $$\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}.$$

CHAPTER VII

THE GAMMA AND BETA FUNCTIONS

67. The Gamma function. By application of the tests of § 62 and § 64 it is easy to show that the integral

$$\int_0^\infty x^{n-1}e^{-x}\,dx$$

converges when n is positive and therefore defines a function of n for positive n. This function is called the Gamma function, and we have

$$\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx. \quad (n>0) \tag{1}$$

We have, directly, $\Gamma(1)=\int_0^\infty e^{-x}\,dx=1.$ (2)

By integration by parts we have the identity

$$\int_0^\infty x^n e^{-x}\,dx=\Big[-x^n e^{-x}\Big]_0^\infty+n\int_0^\infty x^{n-1}e^{-x}\,dx=n\int_0^\infty x^{n-1}e^{-x}\,dx,$$

and therefore $\Gamma(n+1)=n\Gamma(n),$ (3)

which is the fundamental formula for the manipulation of Gamma functions.

It is evident that if the value of $\Gamma(n)$ is known for n between any two successive integers, say between $n=1$ and $n=2$, the value of $\Gamma(n)$ for any positive n may be found by successive applications of (3). Tables for $\log\Gamma(n)$ for $1<n<2$ have been computed and may be found in various places.*

Moreover, formula (3) may be used to define $\Gamma(n)$ for values of n for which the definition (1) fails. For if we write (3) in the form

$$\Gamma(n)=\frac{\Gamma(n+1)}{n}, \tag{4}$$

then, if $-1<n<0$, formula (4) gives us $\Gamma(n)$ since $(n+1)$ is positive. We may then find $\Gamma(n)$ when $-2<n<-1$, since now $\Gamma(n+1)$, in the right hand of (4), is known; and so on indefinitely.

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* Consult, for example, B. O. Peirce's "Short Table of Integrals."

We have, then, in (1) *and* (3) *the complete definition of* $\Gamma(n)$ *for all values of* n.

From (3) it follows that

$$\Gamma(n) = (n-1)\Gamma(n-1); \tag{5}$$

and hence, from (3) and (5),

$$\Gamma(n+1) = n(n-1)\Gamma(n-1),$$

or, more generally,

$$\Gamma(n+1) = n(n-1) \cdots (n-k)\Gamma(n-k), \tag{6}$$

where k is a positive integer.

If n is a positive integer and we take $k = n-1$ in (6), we have, with the aid of (2),

$$\Gamma(n+1) = n! \tag{7}$$

or

$$\Gamma(n) = (n-1)! \tag{8}$$

Accordingly the Gamma function reduces to a factorial number when n is a positive integer, and may therefore be considered as a generalization of $n!$ for the case in which n is fractional or negative.

From (3) we also obtain readily

$$\Gamma(n+k) = (n+k-1) \cdots (n+1)n\Gamma(n), \tag{9}$$

or, what is the same thing,

$$\Gamma(n) = \frac{\Gamma(n+k)}{n(n+1) \cdots (n+k-1)}, \tag{10}$$

where in both (9) and (10) k is a positive integer.

It appears from (10) that the Gamma function becomes infinite when n is zero or a negative integer; for k can be taken large enough in (10) to make $n+k$ positive, and the fraction in (10) then contains a zero factor in the denominator.

The integral (1) may be reduced to other forms, some of which we give, together with the substitution which reduces (1) to each of the new forms:

$$\Gamma(n) = a^n \int_0^\infty y^{n-1} e^{-ay}\, dy, \quad (x = ay) \tag{11}$$

$$\Gamma(n) = 2\int_0^\infty y^{2n-1} e^{-y^2}\, dy, \quad (x = y^2) \tag{12}$$

$$\Gamma(n) = (m+1)^n \int_0^1 y^m \left(\log \frac{1}{y}\right)^{n-1} dy. \quad (x = -(m+1)\log y) \quad (13)$$

If we put $n = \frac{1}{2}$ in (12) and use the result of Example 1, § 65, we get

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \quad (14)$$

68. The Beta function. The integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

converges when m and n are positive (§ 64) and defines a function of m and n called the Beta function. Hence

$$\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx. \quad (1)$$

It is important to notice that m and n may be interchanged in (1). To see this, place $x = 1 - y$. We have

$$\begin{aligned} \mathrm{B}(m, n) &= \int_0^1 (1-y)^{m-1} y^{n-1}\, dy \\ &= \int_0^1 (1-x)^{m-1} x^{n-1}\, dx = \mathrm{B}(n, m). \quad (2) \end{aligned}$$

Other forms of the Beta function are of importance. In (1) place $x = \frac{y}{a}$; then

$$\mathrm{B}(m, n) = \frac{1}{a^{m+n-1}} \int_0^a y^{m-1}(a-y)^{n-1}\, dy. \quad (3)$$

In (1) place $x = \sin^2 \phi$. We get

$$\mathrm{B}(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\phi \cos^{2n-1}\phi\, d\phi. \quad (4)$$

Again, place $x = \frac{y}{1+y}$ in (1). We get

$$\mathrm{B}(m, n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy. \quad (5)$$

A relation between the Beta and Gamma functions may be worked out as follows: Using (12), § 67, we have

$$\begin{aligned} \Gamma(m)\Gamma(n) &= 4\int_0^\infty x^{2n-1} e^{-x^2}\, dx \int_0^\infty y^{2m-1} e^{-y^2}\, dy \\ &= 4\int_0^\infty \int_0^\infty x^{2n-1} y^{2m-1} e^{-(x^2+y^2)}\, dx\, dy. \end{aligned}$$

The double integration is to be taken over the first quadrant of the XOY plane. Replacing Cartesian coördinates by polar coördinates, we have

$$\Gamma(m)\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(m+n-2)}e^{-r^2}\sin^{2m-1}\theta\cos^{2n-1}\phi\, r\, d\theta\, dr$$

$$= 4\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta\int_0^{\infty} r^{2(m+n)-1}e^{-r^2}\, dr$$

$$= \mathrm{B}(m, n)\Gamma(m+n),$$

by (4) of this section and (12), § 67. Hence

$$\mathrm{B}(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{6}$$

As an example, consider $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^{\frac{1}{4}}}}$.

Placing $x^{\frac{1}{4}} = y$, this becomes

$$\int_0^1 4\, y^3(1-y)^{-\frac{1}{2}}\, dy = 4\,\mathrm{B}(4, \tfrac{1}{2})$$

$$= \frac{4\,\Gamma(4)\Gamma(\frac{1}{2})}{\Gamma(\frac{9}{2})};$$

but $\quad \Gamma(4) = 3!, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(\frac{9}{2}) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}.$

Therefore

$$\int_0^1 \frac{dx}{\sqrt{1-x^{\frac{1}{4}}}} = \frac{128}{35}.$$

69. Dirichlet's integrals. As an interesting example of Gamma functions let us endeavor to evaluate the integral

$$I = \iiint x^{l-1}y^{m-1}z^{n-1}\, dx\, dy\, dz$$

over the octant bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and the coördinate planes.

We begin by placing

$$\frac{x^2}{a^2} = \xi, \quad \frac{y^2}{b^2} = \eta, \quad \frac{z^2}{c^2} = \zeta.$$

Then

$$I = \frac{a^l b^m c^n}{8}\iiint \xi^{\frac{l}{2}-1}\eta^{\frac{m}{2}-1}\zeta^{\frac{n}{2}-1}\, d\xi\, d\eta\, d\zeta$$

over the octant bounded by the coördinate planes, and

$$\xi + \eta + \zeta = 1.$$

Putting in limits, we have

$$I = \frac{a^l b^m c^n}{8} \int_0^1 \int_0^{1-\xi} \int_0^{1-\xi-\eta} \xi^{\frac{l}{2}-1} \eta^{\frac{m}{2}-1} \zeta^{\frac{n}{2}-1}\, d\xi\, d\eta\, d\zeta$$

$$= \frac{2\, a^l b^m c^n}{8\, n} \int_0^1 \int_0^{1-\xi} \xi^{\frac{l}{2}-1} \eta^{\frac{m}{2}-1} (1-\xi-\eta)^{\frac{n}{2}}\, d\xi\, d\eta.$$

Carrying out the integration in which ξ is constant, we have, by (3), § 68,

$$I = \frac{a^l b^m c^n}{4\, n} \int_0^1 \xi^{\frac{l}{2}-1} (1-\xi)^{\frac{m+n}{2}} \mathrm{B}\left(\frac{m}{2}, \frac{n}{2}+1\right) d\xi;$$

and carrying out the second integration, we have, by (1) and (6), § 68,

$$I = \frac{a^l b^m c^n}{4\, n} \mathrm{B}\left(\frac{l}{2}, \frac{m+n}{2}+1\right) \mathrm{B}\left(\frac{m}{2}, \frac{n}{2}+1\right)$$

$$= \frac{a^l b^m c^n}{4\, n} \frac{\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m+n}{2}+1\right)}{\Gamma\left(\frac{l+m+n}{2}+1\right)\Gamma\left(\frac{m+n}{2}+1\right)}$$

$$= \frac{a^l b^m c^n}{8} \frac{\Gamma\left(\frac{l}{2}\right)\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{l+m+n}{2}+1\right)}. \tag{1}$$

If $l = 1$, $m = 1$, $n = 1$, we have for the volume of an octant of an ellipsoid

$$V = \frac{abc}{8} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})} = \frac{abc}{6}\pi.$$

If $l = 3$, $m = 1$, $n = 1$, we have

$$\iiint x^2\, dx\, dy\, dz = \frac{a^3bc}{8} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})} = \frac{a^3bc}{30}\pi.$$

Similarly,

$$\iiint y^2\, dx\, dy\, dz = \frac{ab^3c}{30}\pi,$$

and therefore the moment of inertia of an ellipsoid of mass M about OZ is

$$\frac{abc}{30}(a^2+b^2)\pi = \tfrac{1}{5} M(a^2+b^2).$$

Again, if $l = 2$, $m = 2$, $n = 1$, we have the product of inertia

$$I_{xy} = \iiint xy\, dx\, dy\, dz = \frac{a^2b^2c}{8} \frac{\Gamma(1)\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})} = \frac{a^2b^2c}{15}.$$

A more general formula than (1) is

$$\iiint x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{a^l b^m c^n}{pqr}\,\frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)} \qquad (2)$$

over the octant bounded by a portion of the surface

$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$$

and the three coördinate planes. The proof of this we leave to the student.

This discussion may be extended to any number of variables.

70. Special relations. We shall obtain in this section certain relations involving Gamma functions which are less fundamental than those of § 67.

In (5) and (6), § 68, place $m = 1 - n$, assuming that $0 < n < 1$. There results

$$\Gamma(n)\Gamma(1-n) = \int_0^\infty \frac{y^{n-1}}{1+y}\,dy. \qquad (1)$$

We shall show in § 149 that

$$\int_0^\infty \frac{y^{n-1}}{1+y}\,dy = \frac{\pi}{\sin n\pi}. \quad (0 < n < 1) \qquad (2)$$

Assuming this for the present, we have

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}. \qquad (3)$$

We note in passing that by placing $n = \frac{1}{2}$ we have again the result found in § 67,

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \qquad (4)$$

In (3) let us place in succession $n = \frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \cdots, \frac{p-1}{p}$, where p is any positive integer, and multiply the results together. We have

$$\left[\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}\right)\cdots\Gamma\left(\frac{p-1}{p}\right)\right]^2 = \frac{\pi^{p-1}}{\sin\frac{\pi}{p}\sin\frac{2\pi}{p}\cdots\sin\frac{p-1}{p}\pi}.$$

We shall show in § 136 that the denominator on the right is equal to $\frac{p}{2^{p-1}}$. Hence we have

$$\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}\right)\cdots\Gamma\left(\frac{p-1}{p}\right) = \frac{(2\pi)^{\frac{p-1}{2}}}{p^{\frac{1}{2}}}. \qquad (5)$$

To obtain the next relation which we desire we begin with the elementary integral

$$\int_0^\infty e^{-b\alpha}\,d\alpha = \frac{1}{b};$$

whence

$$\int_1^x db \int_0^\infty e^{-b\alpha}\,d\alpha = \int_1^x \frac{db}{b} = \log x.$$

The double integral satisfies the requirements of § 66 for interchange of the order of integration.

Hence

$$\log x = \int_0^\infty d\alpha \int_1^x e^{-b\alpha}\,db = \int_0^\infty \frac{e^{-\alpha}-e^{-\alpha x}}{\alpha}\,d\alpha. \tag{6}$$

The integral (1), § 67, defining $\Gamma(n)$ may be differentiated under the integral sign with respect to n. Representing the result by $\Gamma'(n)$, we have

$$\Gamma'(n) = \int_0^\infty x^{n-1}e^{-x}\log x\,dx;$$

and substituting the value of $\log x$ from (6), we have

$$\Gamma'(n) = \int_0^\infty x^{n-1}e^{-x}\,dx \int_0^\infty \frac{e^{-\alpha}-e^{-\alpha x}}{\alpha}\,d\alpha.$$

The order of integration may be interchanged by § 66. Hence

$$\begin{aligned}\Gamma'(n) &= \int_0^\infty \frac{d\alpha}{\alpha}\int_0^\infty (e^{-\alpha}-e^{-\alpha x})x^{n-1}e^{-x}\,dx \\ &= \int_0^\infty e^{-\alpha}\frac{d\alpha}{\alpha}\int_0^\infty x^{n-1}e^{-x}\,dx - \int_0^\infty \frac{d\alpha}{\alpha}\int_0^\infty x^{n-1}e^{-(\alpha+1)x}\,dx.\end{aligned}$$

The first integrals in each pair may be evaluated by (1), § 67, and (11), § 67, respectively. We have

$$\Gamma'(n) = \Gamma(n)\int_0^\infty \left(e^{-\alpha} - \frac{1}{(1+\alpha)^n}\right)\frac{d\alpha}{\alpha}. \tag{7}$$

By placing $n=1$ in (7), we have

$$\Gamma'(1) = \int_0^\infty \left(e^{-\alpha} - \frac{1}{1+\alpha}\right)\frac{d\alpha}{\alpha},$$

and subtracting this from $\dfrac{\Gamma'(n)}{\Gamma(n)}$, as given in (7), we have

$$\frac{\Gamma'(n)}{\Gamma(n)} = \Gamma'(1) + \int_0^\infty \left[\frac{1}{1+\alpha} - \frac{1}{(1+\alpha)^n}\right]\frac{d\alpha}{\alpha}. \tag{8}$$

If n is an integer greater than unity, this may be reduced further by placing $1+\alpha=t$. The integral in (8) becomes

$$\int_1^\infty [t^{-2}+t^{-3}+\cdots+t^{-n}]dt = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}.$$

The constant $\Gamma'(1)$ is called Euler's constant. It is represented by $-\gamma$, and it has been computed that

$$\gamma = .5772157\cdots.$$

We have, finally, from (8), for integral values of n,

$$\frac{\Gamma'(n)}{\Gamma(n)} = -\gamma+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}. \tag{9}$$

EXERCISES

1. Prove that

$$\Gamma\left(\frac{2k+1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k}\sqrt{\pi},$$

where k is a positive integer.

2. Prove that

$$\int_0^1 x^p(1-x^m)^q\,dx = \frac{\Gamma\left(\frac{p+1}{m}\right)\Gamma(q+1)}{m\Gamma\left(\frac{p+1}{m}+q+1\right)}.$$

3. Prove that $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$

4. Prove that $\displaystyle\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2\pi}}\left[\Gamma\left(\frac{3}{4}\right)\right]^2.$

5. Prove that $\displaystyle\int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)},$

and then show that $\displaystyle\int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}} = \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\frac{\pi}{2}$

if n is an even integer, and

$$\int_0^1 \frac{x^n\,dx}{\sqrt{1-x^2}} = \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}$$

if n is an odd integer.

6. Show that $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n}\,\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}.$

7. From Ex. 2 show that if $\dfrac{p+1}{m}$ is a positive integer k,

$$\int_0^1 x^p(1-x^m)^q\,dx = \frac{1}{m}\,\frac{(k-1)!}{(q+1)(q+2)\cdots(q+k)}.$$

8. From Ex. 2 show that if $q+\dfrac{p+1}{m}$ is a positive integer k,

$$\int_0^1 x^p(1-x^m)^q\,dx = \frac{q(1-q)\cdots(k-1-q)}{m\cdot k\,!\sin q\pi}\,\pi.$$

9. Prove that

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{1}{2}\,\mathrm{B}\left(\frac{n+1}{2}, \frac{1}{2}\right).$$

Hence prove that

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}$$

if n is an odd integer, and

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\,\frac{\pi}{2}$$

if n is an even integer.

10. Prove that

$$\int_0^{\frac{\pi}{2}} \sin^n 2\,x\,dx = \frac{2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2}{\Gamma(n+1)}.$$

11. From Ex. 10 prove that

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2}{\Gamma(n+1)}.$$

12. By combining Exs. 9 and 11 prove that

$$2^{n-1}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}\right) = \sqrt{\pi}\,\Gamma(n).$$

13. Prove that

$$\int_0^\infty \frac{\cos bx}{x^n}\,dx = \frac{b^{n-1}}{\Gamma(n)}\,\frac{\pi}{2\cos\frac{n\pi}{2}} \qquad (0 < n < 1)$$

by placing

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)}\int_0^\infty \alpha^{n-1}e^{-\alpha x}\,d\alpha,$$

reversing the order of integration, and making use of (2), § 70.

14. By a method similar to that of Ex. 13 prove that

$$\int_0^\infty \frac{\sin bx}{x^n}\,dx = \frac{b^{n-1}}{\Gamma(n)}\,\frac{\pi}{2\sin\dfrac{n\pi}{2}}. \qquad (0 < n < 1)$$

15. Find the area of one loop of the curve in polar coördinates

$$r^4 = \sin^3\theta\cos\theta,$$

16. Prove formula (2), § 69.

Use the Dirichlet integrals to obtain the volume, the center of gravity, and the moment of inertia of an octant of each of the solids bounded by the following surfaces:

17. $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$. **18.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$. **19.** $x^2 + y^2 + z^4 = 1$.

CHAPTER VIII

LINE, SURFACE, AND SPACE INTEGRALS

71. Line integrals. Consider a function $P(x, y)$ defined and continuous for a certain region of the plane XOY, and take in the region a curve C (Fig. 61) extending from a point $A(a_1, b_1)$ to a point $B(a_2, b_2)$. Divide the curve C into n segments by points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $\cdots$, let (ξ_i, η_i) be a point in the segment (M_{i-1}, M_i), and form the sum

$$\sum_{i=1}^{i=n-1} P(\xi_i, \eta_i)(x_i - x_{i-1}).$$

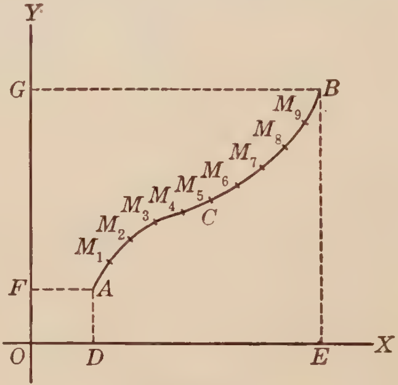

FIG. 61

The limit of this sum as n is indefinitely increased and each factor $x_i - x_{i-1}$ approaches zero is the line integral of $P(x, y)$ along the curve C and is indicated by the symbol

$$\int_{(a_1, b_1)}^{(a_2, b_2)} P(x, y)dx \tag{1}$$

along C. The value of this integral depends not only on the limits but also on the curve C. If the equation of the curve is known in the form

$$y = \phi(x), \tag{2}$$

the integral (1) may be reduced by substitution to an integral in one variable,

$$\int_{a_1}^{a_2} P[x, \phi(x)]dx; \tag{3}$$

or if the equations of the curve are given in terms of a parameter t as

$$x = f_1(t), \quad y = f_2(t), \tag{4}$$

the integral (1) becomes by substitution

$$\int_{t_0}^{t_1} F(t)dt. \tag{5}$$

Either (3) or (5) may be evaluated by direct substitution.

Some very elementary integrals connected with a curve may be looked at from this point of view. For example, $P(x, y)$ may be simply y. Then (1) is the area $ADEB$, of which C is the upper boundary; and if $P(x, y)$ is πy^2, then (1) is the volume of a solid found by revolving $ADEB$ around OX.

If $Q(x, y)$ is another continuous function, we may form the sum

$$\sum_{i=1}^{i=n-1} Q(\xi_i, \eta_i)(y_i - y_{i-1});$$

and the limit of this sum is another line integral,

$$\int_{(a_1, b_1)}^{(a_2, b_2)} Q(x, y)dy, \tag{6}$$

taken along C. Thus, if $Q(x, y)$ is x, the integral (6) is the area $FABG$, of which C is part of the boundary; and if Q is πx^2, the integral (6) is the volume of a solid formed by revolving $FABG$ around OY.

In practice it is more common to find line integrals occurring in the form

$$\int_{(a_1, b_1)}^{(a_2, b_2)} [P(x, y)dx + Q(x, y)dy], \tag{7}$$

which means the sum of (1) and (6). It is this form which we shall generally have in mind when we speak of a line integral. The evaluation of (7) when the equation of C is given in the form (2) or (4) is made by substitution and direct integration.

It is not necessary that the curve C should have the same equation for its entire path, but it may be of the form noticed in § 1. For example, consider the integral

$$\int_{(0, 0)}^{(1, 1)} [(y - x)dy + y\,dx].$$

The curve C may be the parabola $y^2 = x$. Then the integral is

$$\int_0^1 (y + y^2)dy = \tfrac{5}{6}.$$

Or the curve C may consist of a piece of the axis of x from (0, 0) to (1, 0) and then the line $x = 1$ from (1, 0) to (1, 1). The integral is then

$$\int_0^1 (y - 1)dy = -\tfrac{1}{2},$$

since on the first part of the path $y = 0$, $dy = 0$, and on the second portion $x = 1$, $dx = 0$.

We are, however, interested in properties of (7) which may be discussed without direct integration. By the definition, (7) has a distinct meaning as a limit of a sum whether C be an open or a closed curve. In the latter case we write (7) as

$$\int_{\bigcirc}(P\,dx+Q\,dy).$$

As an example of (7) consider a field of force; that is, let there be a force F determined in direction and magnitude at every point of the region of the plane considered.

We wish to find the work done on a particle moving from A to B along a curve C (Fig. 62). Let C be divided into segments each of which is Δs and one of which is MN. Let F be the force at M, MR the direction in which it acts, MT the tangent at M, and θ the angle RMT. Then the component of force in the direction MN is $F\cos\theta$, and the work done on a particle moving from M to N is $F\cos\theta\,ds$ except for infinitesimals of higher order. The total work W in moving the particle from A to B is then

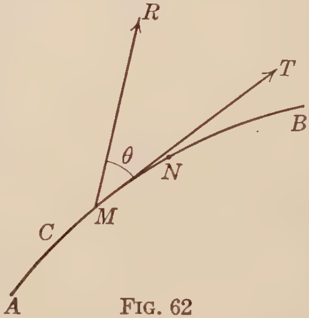

FIG. 62

$$W=\int F\cos\theta\,ds \tag{8}$$

taken along C. This is properly called a line integral, but it is not in the form (7). It may, however, be put in that form. For if α is the angle between MR and OX, and ϕ the angle between MT and OX, then

$$\phi=\alpha-\theta;$$

whence
$$W=\int(F\cos\alpha\cos\phi+F\sin\alpha\sin\phi)ds.$$

But $F\cos\alpha=X$, $F\sin\alpha=Y$, where X and Y are the components of force parallel to OX and OY, respectively, and $\cos\phi\,ds=dx$, $\sin\phi\,ds=dy$.

Therefore
$$W=\int(X\,dx+Y\,dy), \tag{9}$$
which is the form (7).

As another illustration, suppose a fluid flowing over the plane XOY, the lines of flow being all parallel to XOY. We wish to find the amount of fluid per unit of time which flows across a curve C (Fig. 63). Let q be the velocity of the fluid, α the angle which the direction of its motion at each point makes with OX, $u = q \cos \alpha$ the component of velocity parallel to OX, and $v = q \sin \alpha$ the component of velocity parallel to OY. Take an element of the curve $MN = ds$ and neglect all infinitesimals of higher order than ds. In the time dt the element has been transferred to $M'N'$, where $MM' = NN' = q\,dt$. The amount of fluid crossing MN is therefore the amount in a cylinder of base $MM'N'N$. The volume of this cylinder is $hMM'N'N \sin \theta = hq\,dt \sin \theta\,ds$, where h is the depth of the liquid, and θ the angle between MM' and MN. The amount of fluid crossing MN is therefore $h\rho q\,dt \sin \theta\,ds$, where ρ is the density. Hence the total amount crossing MN is

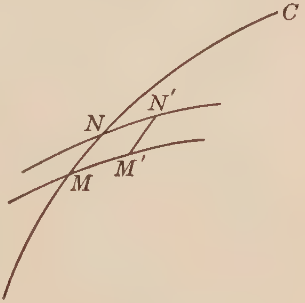

FIG. 63

$$h\,dt \int \rho q \sin \theta\,ds.$$

To put the integral in the form (7), note that if ϕ is the angle made by MN with OX, $\theta = \phi - \alpha$. Therefore the amount flowing across C per unit of time is

$$h \int (-v\rho\,dx + u\rho\,dy), \tag{10}$$

where the integral is of the form (7).

72. Plane area as a line integral. In using line integrals around closed curves we need some method of distinguishing between the two directions in which the curve may be traversed. Accordingly, when the curve is part of the boundary of a specified region, we shall say that the positive direction is that in which a person walking around the curve has the region on his left hand. Thus, if a circle is the boundary of the region included within it, the positive direction is counterclockwise; but if the same circle is part of the boundary of region exterior to it, as in Fig. 64, the positive direction is now clockwise.

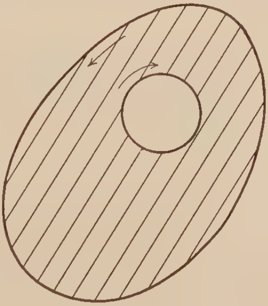
FIG. 64

With this fixed, let us now consider the integral

$$\oint y\,dx, \tag{1}$$

taken in positive direction along a closed curve C which bounds a region of area A (Fig. 65). For simplicity we shall assume first that C is such that a line parallel to OY or OX meets it in two points or not at all, with the exception of four tangents which are parallel to OX or OY at points L, K, M, and N. Here L is the extreme left-hand point of the curve, K the extreme right-hand point, M the highest point, and N the lowest point. We draw the tangents LF, KB, MD, and NE. The integral (1), taken along C from L through N to K, gives the area $FLNKB$. The integral (1), taken along C from K through M to L, gives in magnitude the area $FLMKB$, but with a negative sign, since dx is always negative. Hence (1) taken in the positive direction around C gives the algebraic sum of these areas; namely, the area bounded by C with a negative sign. That is,

$$A = -\int_{\circ} y\,dx. \tag{2}$$

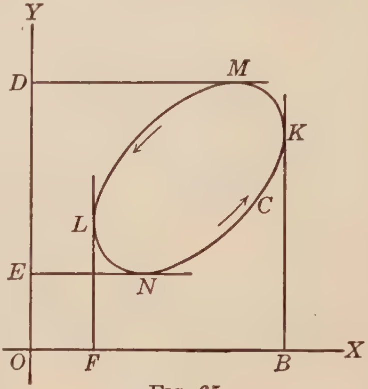

FIG. 65

Consider in a similar manner the integral

$$\int_{\circ} x\,dy. \tag{3}$$

The integral (3), taken along C from N through K to M, gives the area $ENKMD$. The integral (3), taken along C from M through L to N, gives the area $ENLMD$ with a negative sign. Hence

$$A = \int_{\circ} x\,dy. \tag{4}$$

By adding (2) and (4) we have

$$A = \tfrac{1}{2}\int_{\circ} (x\,dy - y\,dx), \tag{5}$$

which expresses the area in terms of a line integral taken around the boundary of the area.

Formula (5) has been proved for an area of simple type. It is readily shown to be true for any area which can be cut up into areas of this type. For example, consider the area bounded by the curve C (Fig. 66). By drawing the lines MN and LK the area is divided into three areas A_1, A_2, A_3, to each of which formula (5) applies. By adding these results we have the area A. But examination of the arrows shows that the curves MN and LK have each been traversed twice, in opposite directions. The integrals along these lines therefore cancel, leaving only the integral (5) around C, traversed continuously in the positive direction.

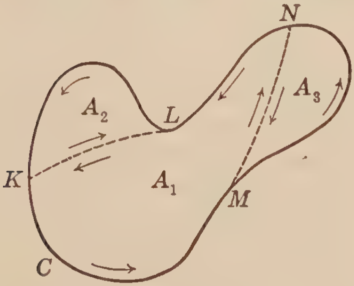

FIG. 66

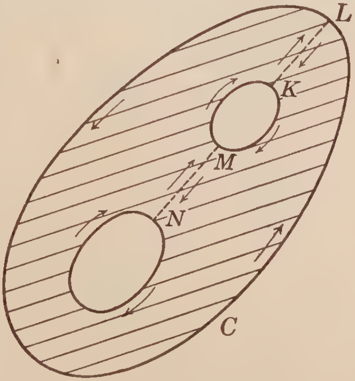

FIG. 67

The theorem is also true for an area bounded by more than one curve. For example, consider Fig. 67. By drawing LK and MN the area is turned into one bounded by a single curve, and formula (5) is applied. The two integrations along LK and MN, however, cancel, leaving the integrals around the boundary curves each traversed in a positive direction.

We have also in the proof assumed that C did not cut OX or OY. The student may easily show that this is immaterial.

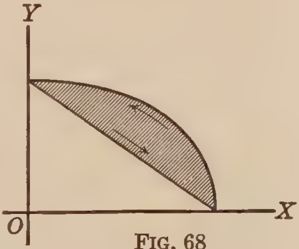

FIG. 68

As an example let us first consider the area bounded by an ellipse and the chord connecting the ends of the major and minor axes (Fig. 68).

The equation of the ellipse is

$$x = a \cos \phi, \quad y = b \sin \phi,$$

so that on the ellipse $\quad x\,dy - y\,dx = ab\,d\phi.$

The equation of the chord is

$$y = -\frac{b}{a}x + b,$$

so that on the chord, $x\,dy - y\,dx = -b\,dx$.

Hence

$$A = \frac{1}{2}\int_{\circ}(x\,dy - y\,dx) = \frac{1}{2}\int_0^{\frac{\pi}{2}} ab\,d\phi + \frac{1}{2}\int_0^a(-b\,dx) = \frac{\pi ab}{4} - \frac{ab}{2}.$$

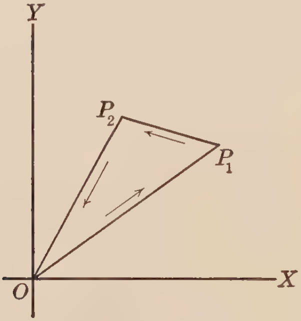

FIG. 69

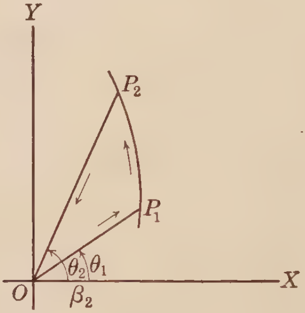

FIG. 70

Again, consider the area of the triangle OP_1P_2 (Fig. 69).

The equation of OP_1 is $y = \frac{y_1}{x_1}x$, so that along OP_1, $x\,dy - y\,dx = 0$. Similarly, along OP_2, $x\,dy - y\,dx = 0$. The equation of P_1P_2 is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

so that along P_1P_2,

$$x\,dy - y\,dx = \left[y_1 - \frac{x_1(y_2 - y_1)}{x_2 - x_1}\right]dx.$$

Hence

$$A = \frac{1}{2}\int_{\circ}(x\,dy - y\,dx) = \frac{1}{2}\int_{x_1}^{x_2}\left[y_1 - \frac{x_1(y_2 - y_1)}{x_2 - x_1}\right]dx = \frac{x_2y_1 - x_1y_2}{2}.$$

Finally, consider the area of the figure OP_1P_2 (Fig. 70) when the curve P_1P_2 is given in polar coördinates. Along this curve

$$x = r\cos\theta, \quad dx = \cos\theta\,dr - r\sin\theta\,d\theta,$$

$$y = r\sin\theta, \quad dy = \sin\theta\,dr + r\cos\theta\,d\theta;$$

whence

$$x\,dy - y\,dx = r^2\,d\theta.$$

Along the lines OP_1 and OP_2 we have, as shown in the previous example, $x\,dy - y\,dx = 0$. Therefore

$$A = \tfrac{1}{2}\int_{O}(x\,dy - y\,dx) = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta,$$

the familiar formula of polar coördinates.

73. Green's theorem in the plane. Consider a region R bounded by a curve C (Fig. 71). We shall assume for convenience that any line drawn through R meets in C in two and only two points. If the line is parallel to OY, one of these, for which $y = y_1$, is on the lower boundary of R, and the other, for which $y = y_2$, is on the upper boundary. Let a and b be the extreme values of x for points in R.

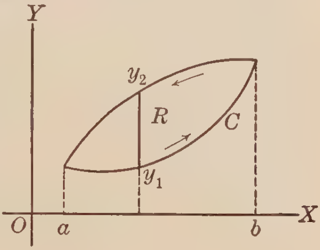

FIG. 71

Let $P(x, y)$ be any function which is continuous in R and on C and for which $\frac{\partial P}{\partial y}$ is continuous. We shall consider the double integral of $\frac{\partial P}{\partial y}$ over the area R. Then, by (2), § 66,

$$\iint\limits_{(R)} \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b dx\int_{y_1}^{y_2}\frac{\partial P}{\partial y}\,dy$$

$$= \int_a^b [P(x, y_2) - P(x, y_1)]dy$$

$$= -\int_a^b P(x, y_1)dy - \int_b^a P(x, y_2)dy. \qquad (1)$$

But by the definition of a line integral the expression on the right in (1) is exactly the line integral of $P\,dy$ around C in the positive direction.

Hence we have $$\iint\limits_{(R)} \frac{\partial P}{\partial y}\,dx\,dy = -\int\limits_{(C)} P\,dx, \qquad (2)$$

where the indices R and C are used to denote the region and the curve over which the integrals are to be taken, and where the direction along C is to be positive.

Similarly, if Q is another function of x and y continuous in R and on C, and such that $\frac{\partial Q}{\partial x}$ is continuous in R, we may show that

$$\iint\limits_{(R)} \frac{\partial Q}{\partial x}\,dx\,dy = \int\limits_{(C)} Q\,dy. \qquad (3)$$

By continuation of (2) and (3) we have, finally,

$$\iint_{(R)} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\, dy = -\int_{(C)} (P\, dx + Q\, dy). \tag{4}$$

We have proved this result for a simple region R. It is easily extended, as in § 72, to regions of more complicated form. This is the first form of Green's theorem. Modifications of (4) follow.

Let α be the angle made with OX by the positive direction of C, and β the angle made with OX by the normal drawn outward.

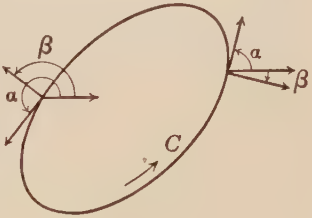

FIG. 72

Then, as shown in Fig. 72, for any point of the curve

$$\alpha = \frac{\pi}{2} + \beta,$$

and

$$\frac{dx}{ds} = \cos\alpha = -\sin\beta,$$

$$\frac{dy}{ds} = \sin\alpha = \cos\beta.$$

If F is a function of x and y, its derivative in the direction along the outward normal to C is, by § 35,

$$\frac{dF}{dn} = \frac{\partial F}{\partial x}\cos\beta + \frac{\partial F}{\partial y}\sin\beta = \frac{\partial F}{\partial x}\frac{dy}{ds} - \frac{\partial F}{\partial y}\frac{dx}{ds}. \tag{5}$$

In (4) we may put $P = -\dfrac{\partial F}{\partial y}$, $Q = \dfrac{\partial F}{\partial x}$,

and have
$$\iint_{(R)} \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) dx\, dy = \int_{(C)} \left(\frac{\partial F}{\partial x}\, dy - \frac{\partial F}{\partial y}\, dx\right); \tag{6}$$

whence, by (5),
$$\int_{(C)} \frac{dF}{dn}\, ds = \iint_{(R)} \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) dx\, dy. \tag{7}$$

Again, if we place in (4)

$$P = -G\frac{\partial F}{\partial y}, \quad Q = G\frac{\partial F}{\partial x},$$

we get
$$\int_{(C)} G\frac{dF}{dn}\, ds = \iint_{(R)} \left[\frac{\partial G}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\frac{\partial F}{\partial y}\right] dx\, dy$$
$$+ \iint_{(R)} G\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) dx\, dy. \tag{8}$$

The equations have special interest when F is a function which satisfies Laplace's equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0.$$

In that case (7) becomes
$$\int_{(C)} \frac{dF}{dn}\, ds = 0. \tag{9}$$

This discussion will be continued in connection with the treatment of Laplace's equation.

74. Dependence upon the path of integration. Consider a region R which shall have the property that any curve in it connecting two points may be gradually deformed into any other curve connecting the same two points without passing out of the region. Such a region is called *simply connected.* A region shown in Fig. 67 is evidently not simply connected until the cuts NM and LK are made, when it becomes simply connected.

We are to inquire under what condition a line integral connecting any two points A and B of such a region depends only on these two points and not on the curve which connects them. It is evident in the first place that if the line integral along the curve C_1 from A to B (Fig. 73) is equal to that along C_2 from A to B, then the integral along the closed curve which we may form by going from A to B along C_1 and from B to A along C_2 is zero. Hence, since the points A and B may be any two points and the curves C_1 and C_2 any two curves, the statement that the line integral between two points is independent of the path is equivalent to the statement that the line integral around any closed curve is zero.

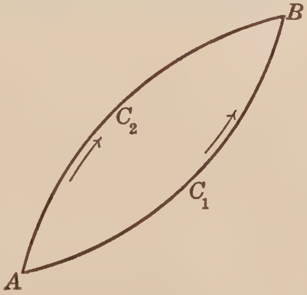

FIG. 73

It is evident that in a simply connected region any closed curve bounds a region to which the results of § 73 may be applied. That is, if $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are single-valued and continuous in R, then for any closed path C inclosing a region T,

$$\int_{(C)} (P\,dx + Q\,dy) = -\iint_{(T)} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\, dy. \tag{1}$$

It is first evident that if $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$ at all points of R, then the integral on the left of (1) is zero. This is then a sufficient condition that a line integral around any closed path in R is zero.

The condition is also necessary, for suppose $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ were not zero at some point. Then it would be possible to take T so that $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ would be of the same sign at all points in T, since the functions concerned are by hypothesis continuous. Hence the integral on the left of (1) could not be zero. Therefore we have the following theorem:

In a simply connected region in which P, Q, and their first partial derivatives are continuous, the necessary and sufficient condition that the integral

$$\int (P\,dx + Q\,dy)$$

around a closed path should be zero and that the integral along a path connecting two points should be independent of the path is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

As an example, consider

$$\int \left[\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right].$$

Here $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$, and the condition is met. But P and Q and their derivatives are discontinuous at the origin, and hence the theorem may be applied only to simply connected regions which do not contain the origin. In other words, the integrals along any two paths which do not inclose the origin is the same, but the integral around two paths which do inclose the origin is not necessarily the same. This may be verified directly. For if we use polar coördinates the integral becomes $\int d\theta$, and θ returns to its original value after traversing a path which does not surround the origin but is increased by a multiple of 2π for any path surrounding the origin.

75. Exact differentials. We have already seen, in § 36, that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1}$$

is the necessary and sufficient condition that

$$P\,dx + Q\,dy$$

is the exact differential of some function ϕ. By the use of the line integral we may establish this result independently of § 36. In the first place, the condition (1) is necessary, since if the function exists such that

$$d\phi = P\,dx + Q\,dy, \tag{2}$$

then

$$\frac{\partial P}{\partial y} = \frac{\partial^2 \phi}{\partial x\, \partial y} = \frac{\partial Q}{\partial x}.$$

We shall proceed to show that, conversely, if (1) is satisfied for any two functions P and Q, there exists a function ϕ for which

$$\frac{\partial \phi}{\partial x} = P, \qquad \frac{\partial \phi}{\partial y} = Q,$$

and therefore

$$P\,dx + Q\,dy = d\phi.$$

Consider for that purpose the integral

$$\int_{(x_0,\, y_0)}^{(x,\, y)} (P\,dx + Q\,dy), \tag{3}$$

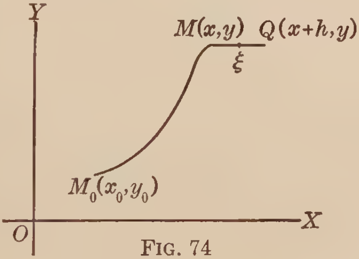

FIG. 74

where (x_0, y_0) is a fixed point M_0 and (x, y) is a variable point M. Under the assumption that (1) is fulfilled, (3) is independent of the path, and its value is therefore determined when (x, y) is given. Hence, by the definition of a function, we are justified in writing

$$\int_{(x_0,\, y_0)}^{(x,\, y)} (P\,dx + Q\,dy) = \phi(x, y). \tag{4}$$

Then

$$\int_{(x_0,\, y_0)}^{(x+h,\, y)} (P\,dx + Q\,dy) = \phi(x + h, y). \tag{5}$$

Since this integral is independent of the path, we may take the path as made up of a curve drawn to M and a line MQ parallel to OX from M to Q (Fig. 74). Then

$$\phi(x + h, y) = \phi(x, y) + \int_{(x,\, y)}^{(x+h,\, y)} (P\,dx + Q\,dy). \tag{6}$$

In the last integral of (6) y is constant. Hence $dy = 0$ and (6) becomes

$$\phi(x + h, y) - \phi(x, y) = \int_{(x, y)}^{(x+h, y)} P(x, y)dx$$
$$= hP(\xi, y), \quad (x < \xi < x + h)$$

the last transformation being made by the theorem of the mean.

Then $$\lim_{h \to 0} \frac{\phi(x + h, y) - \phi(x, y)}{h} = \lim_{\xi \to x} P(\xi, y);$$

whence $$\frac{\partial \phi}{\partial x} = P(x, y).$$

Similarly, we may prove that

$$\frac{\partial \phi}{\partial y} = Q(x, y).$$

We have therefore proved the theorem. The discussion also suggests a method for determining ϕ. Since ϕ as defined by (4) is independent of the path, we may take the path as composed of M_0R (Fig. 75), parallel to OX, along which $P\,dx + Q\,dy = P(x, y_0)dx$, and the line RM, along which $P\,dx + Q\,dy = Q(x, y)dy$ with x constant. We have, therefore,

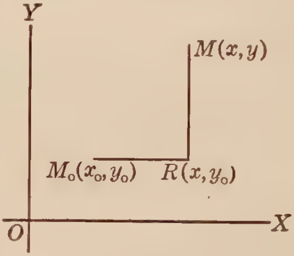

FIG. 75

$$\phi(x, y) = \int_{x_0}^{x} P(x, y_0)dx + \int_{y_0}^{y} Q(x, y)dy, \tag{7}$$

where in the last integral x is constant.

Similarly, by first integrating parallel to OY and then parallel to OX, we have

$$\phi(x, y) = \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy, \tag{8}$$

where in the first integration y is constant.

The point (x_0, y_0) may be chosen in a manner to make the integration as simple as possible, since a change in (x_0, y_0) simply changes the constant of integration. Of course a point (x_0, y_0) must be avoided which would make P or Q infinite or discontinuous along the path of integration.

As an example, consider

$$(4\,x^3 + 10\,xy^3 - 3\,y^4)dx + (15\,x^2y^2 - 12\,xy^3 + 5\,y^4)dy.$$

Here $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=30\,xy^2-12\,y^3$. We may take (x_0, y_0) as $(0, 0)$, and, using (7),

$$\begin{aligned}\phi(x, y) &= \int_0^x 4\,x^3\,dx + \int_0^y (15\,x^2y^2 - 12\,xy^3 + 5\,y^4)dy \\ &= x^4 + 5\,x^2y^3 - 3\,xy^4 + y^5.\end{aligned}$$

Again, consider

$$\left(\frac{1}{x}-\frac{y}{x\sqrt{y^2-x^2}}\right)dx+\frac{1}{\sqrt{y^2-x^2}}\,dy.$$

Here
$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=\frac{x}{(y^2-x^2)^{\frac{3}{2}}}.$$

We cannot take x_0 as zero, since P is then infinite. We may, however, take (x_0, y_0) as $(1, 0)$, and, using (7),

$$\begin{aligned}\phi(x, y) &= \int_1^x \frac{1}{x}\,dx + \int_0^y \frac{dy}{\sqrt{y^2-x^2}} \\ &= \log x + \log\left(y+\sqrt{y^2-x^2}\right) - \log x\sqrt{-1} \\ &= \log\left(y+\sqrt{y^2-x^2}\right) - \log\sqrt{-1}.\end{aligned}$$

Here $\log\sqrt{-1}$ is a constant (§ 140) which may be dropped in writing $\phi(x, y)$. As a check take (x_0, y_0) as $(1, 1)$. Then

$$\begin{aligned}\phi(x, y) &= \int_1^x \left(\frac{1}{x}-\frac{1}{x\sqrt{1-x^2}}\right)dx + \int_1^y \frac{dy}{\sqrt{y^2-x^2}} \\ &= \log x + \log\frac{1+\sqrt{1-x^2}}{x} + \log\left(y+\sqrt{y^2-x^2}\right) \\ &\qquad - \log\left(1+\sqrt{1-x^2}\right) \\ &= \log\left(y+\sqrt{y^2-x^2}\right).\end{aligned}$$

76. Area of a curved surface. We have seen in § 53 that if a surface is defined by the equations

$$\begin{aligned}x &= f_1(u, v), \\ y &= f_2(u, v), \\ z &= f_3(u, v),\end{aligned} \tag{1}$$

the element of area may be taken as

$$dS = \sqrt{EG - F^2}\,du\,dv. \tag{2}$$

In fact, we *define* the area of any portion of the surface (1) as the integral

$$S = \iint \sqrt{EG - F^2}\,du\,dv, \tag{3}$$

the integration to be extended over the portion of the surface considered.

We have laid down somewhat arbitrarily the definition of a curved area. Its validity will depend upon the fact that the number obtained for S is independent of the coördinate system used. This may be shown to be true, but we shall omit the proof. It may be shown without difficulty by the student that our definition gives the usual result for elementary surfaces the areas of which have been found by other methods. For example, the equations

$$x = a \cos \theta \sin \phi,$$
$$y = a \sin \theta \sin \phi,$$
$$z = a \cos \phi,$$

define a sphere of radius a. The entire surface of the sphere is obtained by allowing ϕ to vary from 0 to π, and θ to vary from 0 to 2π. Hence, if $\theta = u$ and $\phi = v$,

$$E = a^2 \sin^2 \phi, \qquad F = 0, \qquad G = a^2,$$

and
$$S = \int_0^{2\pi}\int_0^{\pi} a^2 \sin \phi \, d\theta \, d\phi = 4\pi a^2.$$

A particular case of equations (1) is

$$x = u,$$
$$y = v, \tag{4}$$
$$z = f(u, v) = f(x, y).$$

In this case the coördinate curves on the surface are the curves cut out by planes parallel to OZ and making elements of area $dx\,dy$ on the XOY plane. If we place, as usual, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, we have

$$E = 1 + p^2, \qquad F = pq, \qquad G = 1 + q^2;$$

whence
$$dS = \sqrt{1 + p^2 + q^2}\, dx\, dy. \tag{5}$$

From § 47, if α, β, γ are the angles made with the axes of x, y, z by the normal to the surface at the point (x, y, z), we have

$$\cos \gamma = \frac{1}{\sqrt{1 + p^2 + q^2}},$$

and (3) becomes
$$dS = \sec \gamma \, dx\, dy,$$

or
$$dx\, dy = \cos \gamma \, dS. \tag{6}$$

From this it appears that dS is rigorously the portion of the tangent plane which projects upon the rectangle $dx\,dy$, so that the area of the surface is considered as the limit of the sum of the areas of such portions of tangent planes.

Similarly, in the case

$$x = u,$$
$$y = f(u, v) = f(x, z),$$
$$z = v,$$

we have
$$dS = \sqrt{1 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2}\, dz\, dx;$$

whence
$$dz\, dx = \cos\beta\, dS, \tag{7}$$

where dS projects upon the plane XOZ into the rectangle $dz\, dx$.

Again, if
$$x = f(u, v) = f(y, z),$$
$$y = u,$$
$$z = v,$$
$$dS = \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz;$$

whence
$$dy\, dz = \cos\alpha\, dS. \tag{8}$$

In the use of formulas (6), (7), and (8) it is convenient to consider dS as always positive. Then the projections $dx\, dy$, $dy\, dz$, or $dz\, dx$ will be positive or negative according to the sign of the cosine factor.

Consider, for example, the sphere

$$x^2 + y^2 + z^2 = a^2.$$

If the normal is always drawn outward from a point on the sphere, and γ is the angle between this normal and the positive direction of OZ, then in the use of (6) the projection of the upper hemisphere is positive and of the lower hemisphere is negative. Consequently, for the upper portion of the sphere we have, by the use of (5),

$$dS = \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy. \tag{9}$$

If we wish the area cut out of the upper part of the sphere by the cylinder
$$x^2 + y^2 - ax = 0,$$

we have to compute the integral

$$S = \int_0^a \int_{-\sqrt{ax - x^2}}^{\sqrt{ax - x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy.$$

This is best done by using polar coördinates in the plane. Then

$$S = 2\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{ar\,d\theta\,dr}{\sqrt{a^2 - r^2}} = a^2(\pi - 2).$$

77. Surface integrals. Let F be a function which is defined for each point of a surface S; then

$$\int F\,dS \tag{1}$$

is a surface integral, the summation taking place over the surface.

Here F may be given as a function of (u, v), the curvilinear coördinates on S. Then (1) may be written

$$\iint F(u, v)\sqrt{EG - F^2}\,du\,dv. \tag{2}$$

Or F may be given in terms of (x, y, z), and its value on the surface then determined by the equation of the surface. This gives various forms of the surface integral. For example, we may have

$$\iint F(x, y, z)\cos\gamma\,dx\,dy, \tag{3}$$

where z and $\cos\gamma$ are to be computed from the equation of the surface S. In (3) we may place

$$F(x, y, z)\cos\gamma = R(x, y, z)$$

and have the form

$$\iint R(x, y, z)dx\,dy. \tag{4}$$

In (3) or (4) $dx\,dy$ is to be taken positive or negative according to the law of projection given in § 76; that is, according as $\cos\gamma$ in (3) is positive or negative.

Again, we have as surface integrals on S

$$\iint P(x, y, z)dy\,dz, \tag{5}$$

$$\iint Q(x, y, z)dz\,dx, \tag{6}$$

where $dy\,dz = \cos\alpha\,dS$, $dz\,dx = \cos\beta\,dS$, and the signs of the differentials are to be determined as in § 76.

In applications surface integrals frequently appear as the sum of the three integrals (4), (5), and (6); namely,

$$\iint (P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy), \tag{7}$$

which is the same as

$$\iint (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS. \tag{8}$$

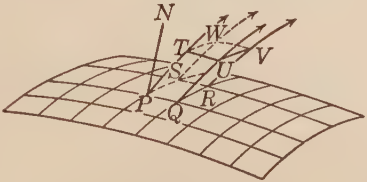

FIG. 76

As an example, suppose fluid flowing across the surface S with a velocity v. Let $PQRS$ be an element of the surface dS and let the lines of flow be as indicated in Fig. 76. In the time dt the particle of fluid at P will flow to T, where $PT = v\,dt$. Hence the volume of fluid flowing in the time dt across dS is equal to the volume of the figure $PQRS$-$TUVW$. Considered as an infinitesimal prism this volume is

$$v\,dt\,dS\cos\phi,$$

where ϕ is the angle between PT and the normal PN to the surface. The amount of fluid in this volume is

$$\rho v\,dt\,dS\cos\phi,$$

where ρ is the density. The amount flowing across the entire surface in the time dt is therefore given by the surface integral

$$dt\int \rho v\cos\phi\,dS, \tag{9}$$

where v and $\cos\phi$ are functions defined for each point on S.

The integral (7) may be transformed as follows: By the law of composition of velocities

$$v\cos\phi = v_n = v_x\cos\alpha + v_y\cos\beta + v_z\cos\gamma,$$

where v_n is the component of velocity normal to S; v_x, v_y, v_z are the components of velocity parallel to OX, OY, OZ; and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal to S. Hence the surface integral in (9) may be written

$$dt\int \rho(v_x\cos\alpha + v_y\cos\beta + v_z\cos\gamma)dS, \tag{10}$$

or

$$dt\iint (\rho v_x\,dy\,dz + \rho v_y\,dz\,dx + \rho v_z\,dx\,dy), \tag{11}$$

where the signs of the differentials in (9) must be taken as in § 76.

78. Green's theorem in space. Let R be a function of x, y, and z and consider the integral

$$\iiint \frac{\partial R}{\partial z}\, dx\, dy\, dz \tag{1}$$

taken through a volume T in which R and $\frac{\partial R}{\partial z}$ are continuous.

Let us first suppose the region T bounded by a surface S which projects upon the XOY plane into a region S' such that a straight line drawn parallel to OZ from any point in S' meets S in two and only two points (Fig. 77). Let z_1 and z_2 $(z_1 < z_2)$ be the values of z at these points. Then the integral (1) may be written

$$\iint\limits_{(S')} dx\, dy \int_{z_1}^{z_2} \frac{\partial R}{\partial z}\, dz = \iint\limits_{(S')} [R(x, y, z_2) - R(x, y, z_1)]\, dx\, dy. \tag{2}$$

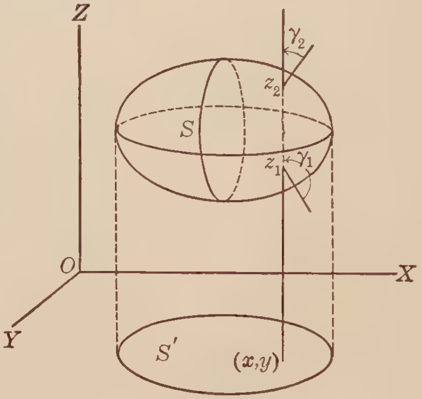

FIG. 77

Let the normal to the surface be drawn outward at each point, let α, β, γ be the usual angles, and let $\gamma = \gamma_1$ when $z = z_1$ and $\gamma = \gamma_2$ when $z = z_2$. Then it is evident that γ_1 is obtuse and that γ_2 is acute. In (2), however, $dx\, dy$ is positive from the nature of the triple integral involved. Hence we have, if dS is the positive element of area on the surface,

$$dx\, dy = -\cos\gamma_1\, dS$$

when $z = z_1$,

$$dx\, dy = \cos\gamma_2\, dS$$

when $z = z_2$,

and then (2) may be written

$$\iint\limits_{(S')} [R(x, y, z_2) + R(x, y, z_1)] \cos\gamma\, dS.$$

But this is simply the surface integral of $R \cos\gamma$ over the surface, and hence we have

$$\iiint\limits_{(T)} \frac{\partial R}{\partial z}\, dx\, dy\, dz = \iint\limits_{(S)} R \cos\gamma\, dS. \tag{3}$$

Finally, in (3) place $\cos\gamma\, dS = dx\, dy$, and we have

$$\iiint\limits_{(T)} \frac{\partial R}{\partial z}\, dx\, dy\, dz = \iint\limits_{(S)} R\, dx\, dy. \tag{4}$$

In (4) $dx\,dy$ is to be given a sign in accordance with the convention of § 76. This makes an essential difference between the right-hand side of equation (4) and the right-hand side of equation (2).

We have assumed for convenience a simple form of the volume T. It is easy to extend the result to volumes which may be split up into volumes of this type. This we leave to the student.

In the same manner we have

$$\iiint\limits_{(T)} \frac{\partial Q}{\partial y}\,dx\,dy\,dz = \iint\limits_{(S)} Q\cos\beta\,dS = \iint\limits_{(S)} Q\,dz\,dx, \tag{5}$$

$$\iiint\limits_{(T)} \frac{\partial P}{\partial x}\,dx\,dy\,dz = \iint\limits_{(S)} P\cos\alpha\,dS = \iint\limits_{(S)} P\,dy\,dz. \tag{6}$$

By adding (4), (5), and (6) we obtain the result which is most often in use:

$$\begin{aligned}\iiint\limits_{(T)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx\,dy\,dz &= \iint\limits_{(S)} (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS \\ &= \iint\limits_{(S)} (P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy). \end{aligned} \tag{7}$$

This is one of the relations which are known as *Green's theorem.* As an illustration of the meaning of (7) let us consider the integral

$$\iint [(x^3 - yz)dy\,dz - 2\,x^2y\,dz\,dx + z\,dx\,dy] \tag{8}$$

taken over a cube of side equal to a, three of whose edges lie along the coördinate axes. Applying (7), this is equal to

$$\int_0^a\int_0^a\int_0^a (3\,x^2 - 2\,x^2 + 1)dx\,dy\,dz = \frac{a^5}{3} + a^3. \tag{9}$$

On the other hand, proceeding directly and considering

$$\iint\limits_{(S)} (x^3 - yz)dy\,dz,$$

we have $dy\,dz$ positive on the face $x = a$, negative on the face $x = 0$, and zero on all other faces. Hence the integral is

$$\int_0^a\int_0^a (a^3 - yz)dy\,dz + \int_0^a\int_0^a yz\,dy\,dz = a^5. \tag{10}$$

In the integral $\displaystyle\iint\limits_{(S)} (-2\,x^2y)dz\,dx$

$dz\,dx$ is positive on the face $y = a$, negative on the face $y = 0$, and zero on all other faces. Hence the integral is

$$\int_0^a \int_0^a (-2\,ax^2)dz\,dx = -\tfrac{2}{3}\,a^5. \tag{11}$$

In the integral
$$\iint\limits_{(S)} z\,dx\,dy$$

$dx\,dy$ is positive on the face $z = a$, negative on the face $z = 0$, and zero on all other faces. Hence its value is

$$\int_0^a \int_0^a a\,dx\,dy = a^3. \tag{12}$$

Combining (10), (11), and (12), we find the result (9).

As an application consider the surface integral as given in (11), § 77, for the amount of fluid flowing out across a surface in the time dt. This must be equal to the loss of fluid in the volume bounded by the surface. The amount of fluid in an element $dx\,dy\,dz$ at the time t is $\rho\,dx\,dy\,dz$ and in the time $t + dt$ it is $(\rho + \frac{\partial \rho}{\partial t}\,dt)dx\,dy\,dz$. The loss in a single element is, then,

$$-\frac{\partial \rho}{\partial t}\,dt\,dx\,dy\,dz,$$

and in the whole volume it is

$$-dt\iiint\limits_{(T)} \frac{\partial \rho}{\partial t}\,dx\,dy\,dz. \tag{13}$$

But the surface integral in (11), § 77, may be transformed by means of (7) into the space integral

$$dt\iiint\limits_{(T)} \left(\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right)dx\,dy\,dz. \tag{14}$$

The two integrals (13) and (14) are equal, and hence we have

$$\iiint\limits_{(T)} \left(\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} + \frac{\partial \rho}{\partial t}\right)dx\,dy\,dz = 0. \tag{15}$$

Now (15) is to be true for any volume T, no matter how small, within which the integral is continuous. Hence the integrand must be everywhere zero, for if it were not, it would be possible to find a volume for which (15) would not be true. We have, therefore,

$$\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} + \frac{\partial \rho}{\partial t} = 0, \tag{16}$$

the so-called *equation of continuity* in hydromechanics.

In the particular case of a liquid for which $\rho =$ constant, equation (16) becomes

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{17}$$

79. Other forms of Green's theorem. Let $F(x, y, z)$ and $G(x, y, z)$ be two functions which are continuous and have continuous first derivatives in the region in which S lies. Then the derivative of F normal to S is, by (10), § 35,

$$\frac{dF}{dn} = \frac{\partial F}{\partial x} \cos \alpha + \frac{\partial F}{\partial y} \cos \beta + \frac{\partial F}{\partial z} \cos \gamma. \tag{1}$$

In (7), § 78, let us place $P = G\frac{\partial F}{\partial x}$, $Q = G\frac{\partial F}{\partial y}$, $R = G\frac{\partial F}{\partial z}$. We obtain, with the use of (1),

$$\iint\limits_{(S)} G\frac{dF}{dn}\,dS = \iiint\limits_{(T)} \left(\frac{\partial G}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\frac{\partial F}{\partial y} + \frac{\partial G}{\partial z}\frac{\partial F}{\partial z}\right) dx\,dy\,dz \\ + \iiint\limits_{(T)} G\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\right) dx\,dy\,dz, \tag{2}$$

another form of Green's theorem.

It is allowable in (2) to place $G = 1$. We then have the simpler form

$$\iint\limits_{(S)} \frac{dF}{dn}\,dS = \iiint\limits_{(T)} \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\right) dx\,dy\,dz. \tag{3}$$

Also, in (2), we may interchange F and G and, subtracting the new result from (2), have

$$\iint\limits_{(S)} \left(G\frac{dF}{dn} - F\frac{dG}{dn}\right) dS = \iiint\limits_{(T)} \left[G\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\right) \right. \\ \left. - F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2}\right)\right] dx\,dy\,dz. \tag{4}$$

The results (2), (3), and (4) take simpler forms if F and G are functions which satisfy the Laplace differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 1.$$

For example, (3) then gives

$$\iint\limits_{(S)} \frac{dF}{dn}\, dS = 0. \tag{5}$$

As an application consider a charge of electricity e at a point (a, b, c). The potential at a point $P(x, y, z)$ at a distance r is

$$F = \frac{e}{r},$$

where $$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

Then for a region T which does not contain (a, b, c) we have formula (5). But if the region T contains (a, b, c) the formula is not applicable, since F is discontinuous when $r = 0$. We will compute

$$\iint \frac{dF}{dn}\, dS$$

directly for a sphere of center (a, b, c) and radius r which we will denote by Σ. We have for this sphere

$$\frac{dF}{dn} = \frac{dF}{dr} = \frac{d}{dr}\left(\frac{e}{r}\right) = -\frac{e}{r^2},$$

where the normal is drawn outward from the sphere, and on the sphere $r = r_0$. Hence

$$\iint\limits_{(\Sigma)} \frac{dF}{dn}\, dS = -\frac{e}{r_0^2}\iint\limits_{(\Sigma)} dS = -4\pi e. \tag{6}$$

Consider any surface S surrounding (a, b, c) and construct a sphere Σ about (a, b, c) as a center. In the region T, bounded by S and Σ, formula (5) applies. Hence

$$\iint\limits_{(S)} \frac{dF}{dn}\, dS + \iint\limits_{(\Sigma)} \frac{dF}{dn}\, dS = 0. \tag{7}$$

In the second integral of this formula the normal must be drawn outward from T and therefore into the sphere Σ. Hence the sign of (6) must be changed, and (7) gives

$$\iint\limits_{(S)} \frac{dF}{dn}\, dS = -4\pi e. \tag{8}$$

In the same manner, if we have n charges of electricity $e_1, e_2, \cdots, e_n$ at points $(a_1, b_1, c_1), (a_2, b_2, c_2), \cdots, (a_n, b_n, c_n)$ inside of S, we find that

$$\iint_{(S)} \frac{dF}{dn}\, dS = -4\,\pi(e_1 + e_2 + \cdots + e_n) = -4\,\pi e; \tag{9}$$

or if we place $N = -\frac{dF}{dn}$, the normal force or intensity perpendicular to S, then

$$\iint_{(S)} N\, dS = 4\,\pi e. \tag{10}$$

We will apply (10) to finding the intensity N due to a spherical conductor of charge e. Take as the surface S a sphere of radius R concentric with the conductor and surrounding it. By symmetry the intensity N is constant on S. Hence (10) gives

$$N \iint_{(S)} dS = 4\,\pi R^2 N = 4\,\pi e;$$

whence

$$N = \frac{e}{R^2}.$$

Again, we will apply (10) to finding the intensity due to an infinite circular cylinder of charge e per unit length. Take as the surface S a circular cylinder of height unity and radius R the axis of which coincides with the axis of the conductor. The charge inside S is then that on a unit length of the conductor, it being assumed that R is greater than the radius of the conductor.

On the upper and lower bases of S we have $N = 0$, since the force is perpendicular to the conductor by symmetry. On the curved surface of S, by symmetry, N is constant. Hence we have, from (10),

$$N \iint_{(S)} dS = 2\,\pi R N = 4\,\pi e;$$

whence

$$N = \frac{2\,e}{R}.$$

80. Stokes's theorem. If P, Q, and R are three functions of x, y, and z, the line integral

$$\int_{(C)} (P\, dx + Q\, dy + R\, dz) \tag{1}$$

along a space curve C is defined in a manner precisely similar to the definition of a line integral along a plane curve.

Let C be a closed curve, and let a surface S be bounded by C. Let the equations of S be in the general form (1), § 53, and for convenience let us write

$$x_u = \frac{\partial x}{\partial u}, \quad x_v = \frac{\partial x}{\partial v}, \quad y_u = \frac{\partial y}{\partial u}, \quad y_v = \frac{\partial y}{\partial v}, \quad z_u = \frac{\partial z}{\partial u}, \quad z_v = \frac{\partial z}{\partial v}.$$

Then we have

$$\int_{(C)} P\,dx = \int_{(C)} (Px_u\,du + Px_v\,dv) \tag{2}$$

$$= -\iint_{(S)} \left[\frac{\partial}{\partial v}(Px_u) - \frac{\partial}{\partial u}(Px_v)\right] du\,dv, \tag{3}$$

since the argument which was used to obtain (4), § 73, may be transferred without change to the surface S with the curvilinear coördinates (u, v). The expression (3) is easily reduced to

$$\int_{(C)} P\,dx = -\iint_{(S)} \left[\frac{\partial P}{\partial y}(x_u y_v - x_v y_u) + \frac{\partial P}{\partial z}(x_u z_v - x_v z_u)\right] du\,dv; \tag{4}$$

whence, by (11) and (9), § 53,

$$\int_{(C)} P\,dx = -\iint_{(S)} \left(\frac{\partial P}{\partial y}\cos\gamma - \frac{\partial P}{\partial z}\cos\beta\right) dS. \tag{5}$$

Similarly, $$\int_{(C)} Q\,dy = -\iint_{(S)} \left(\frac{\partial Q}{\partial z}\cos\alpha - \frac{\partial Q}{\partial x}\cos\gamma\right) dS, \tag{6}$$

and $$\int_{(C)} R\,dz = -\iint_{(S)} \left(\frac{\partial R}{\partial x}\cos\beta - \frac{\partial R}{\partial y}\cos\alpha\right) dS. \tag{7}$$

By adding (5), (6), and (7) we have *Stokes's theorem*; namely,

$$\int_{(C)} (P\,dx + Q\,dy + R\,dz) = -\iint_{(S)} \left[\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\cos\alpha + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\cos\beta + \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\cos\gamma\right] dS. \tag{8}$$

A word about signs is necessary. A change in the direction in which the line integral is taken would change the sign of the left-hand member of (8), and a change in the direction in which the normal to the surface is taken would change the signs of cos α,

$\cos\beta$, $\cos\gamma$ and therefore change the sign of the right-hand member of (8). Hence there must be a relation between the direction of the normal to the surface and the direction of integration around the curve. From the proof it appears that the relation between the normal to the surface and the direction of integration must be the same as the relation between the normal to a plane and the direction of the integration in § 73. Hence an observer standing with his feet on the surface and his head in the direction of the normal will see the integration around C taken in the positive direction.

From equation (8) it follows, by arguments similar to those of § 74, that in a simply connected region of space in which P, Q, R and their derivatives are single-valued and continuous, the necessary and sufficient conditions that the line integral (1) between fixed limits should be independent of the path connecting those limits, and that the same integral around a closed path should be zero, are

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}. \tag{9}$$

If these conditions are met, the line integral

$$\int_{(x_0,\, y_0,\, z_0)}^{(x,\, y,\, z)}(P\,dx+Q\,dy+R\,dz)$$

defines a function $\phi(x, y, z)$ such that

$$P=\frac{\partial\phi}{\partial x},\quad Q=\frac{\partial\phi}{\partial y},\quad R=\frac{\partial\phi}{\partial z}, \tag{10}$$

as is easily shown by the methods of § 75.

Conversely, if there is a function ϕ for which equations (10) are true, then equations (9) follow immediately. Hence equations (9) are the necessary and sufficient conditions that such a function ϕ exists, or, in other words, that $P\,dx+Q\,dy+R\,dz$ is an exact differential, so that we may write

$$P\,dx+Q\,dy+R\,dz=d\phi. \tag{11}$$

Apply this to a field of force in which the components of force at each point are X, Y, Z. By definition the force has a potential V if there is such a function V that

$$\frac{\partial V}{\partial x}=-X,\quad \frac{\partial V}{\partial y}=-Y,\quad \frac{\partial V}{\partial z}=-Z.$$

From the discussion given above the necessary and sufficient conditions for this are

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}, \quad \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}, \quad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z},$$

and we have, then,
$$X\,dx + Y\,dy + Z\,dz = -\,dV.$$

The work done in passing between two points (x_0, y_0, z_0) and (x_1, y_1, z_1) is

$$\int_{(x_0, y_0, z_0)}^{(x_1, y_1, z_1)} (X\,dx + Y\,dy + Z\,dz) = -\int_{(x_0, y_0, z_0)}^{(x_1, y_1, z_1)} dV$$
$$= V(x_0, y_0, z_0) - V(x_1, y_1, z_1).$$

The work done is then independent of the path and equal to the difference of potential between the beginning and the end of the path.

Again, if
$$P = v_x, \quad Q = v_y, \quad R = v_z,$$

are the components of velocity of a liquid, that velocity is said to have a velocity potential ϕ if there exists such a function ϕ that

$$\frac{\partial \phi}{\partial x} = v_x, \quad \frac{\partial \phi}{\partial y} = v_y, \quad \frac{\partial \phi}{\partial z} = v_z.$$

In this case the motion of the liquid is called irrotational, whereas motion without a velocity potential is called vortex motion. The necessary and sufficient conditions for irrotational motion are that v_x, v_y, v_z satisfy the relations

$$\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x}, \quad \frac{\partial v_y}{\partial z} = \frac{\partial v_z}{\partial y}, \quad \frac{\partial v_z}{\partial x} = \frac{\partial v_x}{\partial z}.$$

From (17), § 78, it appears that ϕ must satisfy the partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

In hydromechanics the integral

$$\int (v_x\,dx + v_y\,dy + v_z\,dz)$$

along any path is called the circulation along that path. It appears that for irrotational motion the circulation around a closed path is zero.

EXERCISES

1. Find the value of

$$\int_{(0,\,0)}^{(1,\,2)}[(x+y^2)dx+2\,xy^2\,dy]$$

along the following paths:

(1) A straight line.
(2) A parabola $y^2=4\,x$.
(3) A portion of the x-axis and a straight line perpendicular to it.

2. Find the value of

$$\int_{(0,\,0)}^{(1,\,3)}[y^2\,dx+(xy-x^2)dy]$$

along the following paths:

(1) $y=3\,x$.
(2) $y^2=9\,x$.
(3) A portion of the y-axis and a straight line perpendicular to it.

3. Find the value of

$$\int_{(0,\,2)}^{(2,\,0)}[(x^2-y^2)dx+x\,dy]$$

along the following paths:

(1) A straight line.
(2) A circle with center at O.

4. Find the area of the four-cusped hypocycloid $x=a\cos^3\phi$, $y=a\sin^3\phi$.

5. Find the area between one arch of a hypocycloid and the fixed circle.

6. Find the area between one arch of an epicycloid and the fixed circle.

7. Find the area of the loop of the curve $x=\dfrac{a(1-t^2)}{1+t^2}$, $y=\dfrac{at(1-t^2)}{1+t^2}$ corresponding to values of t between -1 and $+1$.

8. Find the area of the segment of a circle of radius a cut off by a chord b units from the origin.

9. Show that the integral

$$\int_{(0,\,0)}^{(1,\,2)}[3\,x(x+2\,y)dx+(3\,x^2-y^3)dy]$$

is independent of the path, and find its value.

10. Show that $\displaystyle\int_{(1,\,1)}^{(x,\,y)}\left(\frac{2\,x-y}{x^2+y^2}\,dx+\frac{2\,y+x}{x^2+y^2}\,dy\right)$

is independent of the path, and find its value.

11. Show that $\int_{(1,0)}^{(2,1)}\left(\frac{1+y^2}{x^3}\,dx - \frac{1+x^2}{x^2}\,y\,dy\right)$ is independent of the path, and check by evaluating for two different paths.

12. Find the area of the surface cut from the cylinder $y^2+z^2=a^2$ by the cylinder $x^2+y^2=a^2$.

13. Find the area of the surface of a sphere of radius a intercepted by a right circular cylinder of radius $\frac{1}{2}a$ if an element of the cylinder passes through the center of the sphere.

14. Find the area of the surface of the cone $x^2+y^2-z^2=0$ cut out by the cylinder $x^2+y^2-2\,ax=0$.

15. Find the area of that part of the surface $z=\frac{x^2-y^2}{2\,a}$ the projection of which on the plane XOY is bounded by the curve $r^2=a^2\cos 2\,\theta$.

16. Find the area of the surface cut from the paraboloid $y^2+z^2=4\,ax$ by the cylinder $y^2=ax$ and the plane $x=3\,a$.

17. Find the area of the surface of the cone $x^2+y^2-4\,z^2=0$ cut out by the cylinder $x^2+y^2-4\,x=0$.

Apply Green's theorem to the following integrals and verify by direct calculation:

18. $\iint(xz\,dy\,dz+yz\,dz\,dx+z^2\,dx\,dy)$ over the sphere $x^2+y^2+z^2=a^2$.

19. $\iint(x\,dy\,dz+y\,dz\,dx+z\,dx\,dy)$ over the cylinder $x^2+y^2=a^2, z=\pm b$.

20. $\iint(dy\,dz+dz\,dx+dx\,dy)$ over any surface.

21. $\iint(x^2\,dy\,dz+y^2\,dz\,dx+z^2\,dx\,dy)$ over the sphere $x^2+y^2+z^2=a^2$.

22. Compute $\int_{(0,0,0)}^{(1,1,1)}(y^2\,dx+xy\,dy+zx\,dz)$ along the following paths:

(1) A straight line from the lower limit to the upper limit.
(2) A broken line consisting of parts parallel to the axes.

23. Compute $\int_{(0,0,0)}^{(1,1,1)}(yz\,dx+zx\,dy+xy\,dz)$ along the same paths as those given in Ex. 22.

24. Show that

$$\int(P\,dx+Q\,dy+R\,dz)=\int\sqrt{P^2+Q^2+R^2}\cos\theta\,ds,$$

where ds is the element of arc of the curve and θ is the angle between the curve and the direction $P:Q:R$.

25. If F satisfies Laplace's equation, show that

$$\iiint_{(T)}\left[\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2\right]dx\,dy\,dz=\iint_{(S)}F\frac{dF}{dn}\,dS.$$

CHAPTER IX

VECTOR NOTATION

81. Vectors. A vector is a directed magnitude. Examples are force, velocity, acceleration. A vector may be graphically represented by a portion of a straight line with a definite length and a definite direction. The position of the line is unessential. Two vectors are equal if they have the same direction and magnitude, no matter how they may lie in space. One vector is the negative of another if the two have the same length and opposite directions.

A scalar is a magnitude without direction. Examples are temperature, density, potential. The length of a vector is a scalar quantity.

It is customary to represent vectors by Greek letters, α, β, γ, $\cdots$, or blackface $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\cdots$. The length of a vector may then be denoted by $|\alpha|$ or $|\mathbf{a}|$ or the corresponding lightface letter a.

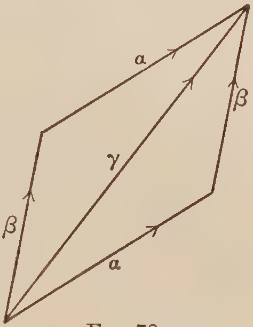

FIG. 78

Two vectors are added by the law of composition of forces or velocities. Thus, if α and β are two vectors, their sum is the diagonal of the parallelogram of which α and β are two sides. Thus, in Fig. 78,

$$\gamma = \alpha + \beta. \tag{1}$$

Any number of vectors may be graphically added by taking them in any order and placing the beginning of each on the end of the preceding. The sum is then the vector which joins the beginning of the first vector to the end of the last vector. Thus, in Fig. 79,

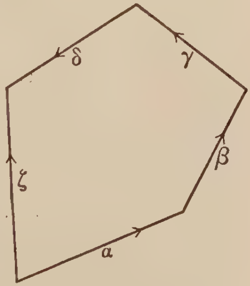

FIG. 79

$$\zeta = \alpha + \beta + \gamma + \delta. \tag{2}$$

Take three mutually perpendicular directions from a point O (Fig. 80). Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the vectors of unit length in these directions. These directions shall be taken so that an observer standing with the vector **k** running from his foot to his head sees the rotation from **i** to **j** as a positive rotation.

Let α be any vector from O whose projections on the three directions are A_1, B_1, and C_1 respectively. Then, from the law of addition,

$$\alpha = A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}, \tag{3}$$

and

$$|\alpha| = \sqrt{A_1^2 + B_1^2 + C_1^2}. \tag{4}$$

Then, if

$$\beta = A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k},$$

it is readily seen that

$$\alpha + \beta = (A_1 + A_2)\mathbf{i} + (B_1 + B_2)\mathbf{j} + (C_1 + C_2)\mathbf{k}. \tag{5}$$

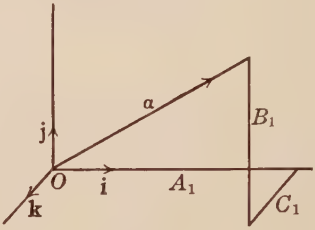

FIG. 80

A vector is multiplied by a positive scalar quantity by multiplying its length by that quantity without changing its direction. It is multiplied by a negative scalar quantity by reversing its direction and multiplying its length by the absolute value of the scalar.

Two vectors may be multiplied in two ways, giving rise to a scalar product or a vector product, to be discussed in the next sections.

82. The scalar product. The scalar product, or the dot product, of two vectors α and β is defined by the equation

$$\alpha\cdot\beta = ab \cos \theta, \tag{1}$$

where a and b are the lengths of α and β respectively, and θ is the angle between them.

In Fig. 81 let OA be the vector α, OB the vector β, ON the projection of β on α, and OM the projection of α on β. Then

$$ab \cos \theta = ON \cdot OA = OM \cdot OB, \tag{2}$$

so that the scalar product of two vectors is the product of the length of either vector by the length of the projection of the other upon it.

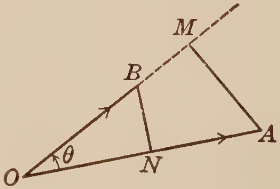

FIG. 81

From (1) we have

$$\beta\cdot\alpha = \alpha\cdot\beta, \tag{3}$$

so that the scalar product is commutative.

Also from the projection property (2) it is evident that

$$\alpha\cdot(\beta + \gamma) = \alpha\cdot\beta + \alpha\cdot\gamma, \tag{4}$$

so that the scalar product is distributive. From (4) it follows that

$$(\alpha + \delta)\cdot(\beta + \gamma) = \alpha\cdot\beta + \alpha\cdot\gamma + \delta\cdot\beta + \delta\cdot\gamma,$$

as in ordinary multiplication, and so for more extended products.

From (1),

$$\alpha\cdot\alpha = a^2, \tag{5}$$

so that the scalar product of a vector by itself is equal to the square of its length.

From (1), if

$$\alpha\cdot\beta = 0,$$

we have either $a = 0$ or $b = 0$ or $\cos\theta = 0$; that is, one of the vectors is of zero length or the two vectors are perpendicular. Assuming that the vectors are not of zero length, we say that *the vanishing of the scalar product of two vectors is the condition for their perpendicularity.*

It follows that "cancellation" as employed in algebraic equations is not legitimate for vectors. That is, if

$$\alpha\cdot\beta = \alpha\cdot\gamma, \tag{6}$$

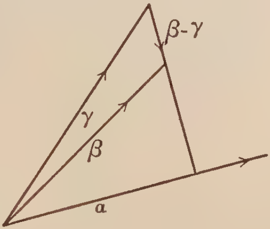

FIG. 82

it does not follow that $\beta = \gamma$. For from (6) we have, by subtracting $\alpha\cdot\gamma$ from each side,

$$\alpha\cdot\beta - \alpha\cdot\gamma = 0,$$

and, by (4),

$$\alpha\cdot(\beta - \gamma) = 0;$$

whence it follows that $\beta - \gamma$ is perpendicular to α. This is graphically shown in Fig. 82, where the projections of γ and β on α are equal.

If **i**, **j**, **k** are the perpendicular unit vectors defined in § 81, then

$$\begin{aligned} \mathbf{i}\cdot\mathbf{i} &= 1, & \mathbf{j}\cdot\mathbf{j} &= 1, & \mathbf{k}\cdot\mathbf{k} &= 1, \\ \mathbf{i}\cdot\mathbf{j} &= 0, & \mathbf{j}\cdot\mathbf{k} &= 0, & \mathbf{k}\cdot\mathbf{i} &= 0. \end{aligned} \tag{7}$$

If

$$\alpha = A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}$$

and

$$\beta = A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k},$$

$\alpha\cdot\beta$ may be computed by the distributive law of multiplication and reduced by (7), with the result

$$\alpha\cdot\beta = A_1A_2 + B_1B_2 + C_1C_2. \tag{8}$$

From (1) of this section and (4) of § 81 we get

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}, \tag{9}$$

and the condition for the perpendicularity of the two vectors is

$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \tag{10}$$

83. The vector product. The vector product, or cross product, of two vectors α and β is defined by the equation

$$\alpha \times \beta = \nu\, ab \sin \theta, \tag{1}$$

where a and b are the lengths of α and β respectively, θ is the angle between them, and ν is a unit vector perpendicular to α and β and in such a direction that an observer standing so that ν runs from his feet to his head sees the rotation from α to β as positive. This is shown in Fig. 83. The length of the vector $\alpha \times \beta$ is in linear units equal to the area in square units of the parallelogram determined by α and β.

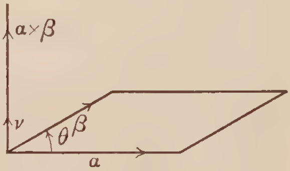

FIG. 83

From the definition we have

$$\alpha \times \alpha = 0, \tag{2}$$

so that the vector product of a vector by itself is zero. In general, if

$$\alpha \times \beta = 0, \tag{3}$$

the vector α is parallel to the vector β, so that *the condition for the parallelism of two vectors is the vanishing of their vector product.*

From the definition it follows that to interchange the order of the factors α and β changes the direction of ν. Hence

$$\beta \times \alpha = -\alpha \times \beta, \tag{4}$$

so that a vector product is not commutative.

Also $\alpha \times (\beta \times \gamma)$ is not equal to $(\alpha \times \beta) \times \gamma$, where α, β, and γ are three vectors in general positions. For the vector $\beta \times \gamma$ is perpendicular to the plane of β and γ, and therefore the vector $\alpha \times (\beta \times \gamma)$ must lie in the plane of β and γ. Similarly, $(\alpha \times \beta) \times \gamma$ lies in the plane of α and β. Hence in general

$$\alpha \times (\beta \times \gamma) \neq (\alpha \times \beta) \times \gamma, \tag{5}$$

so that a vector product is not associative.

It is true, however, that

$$\gamma \times (\alpha + \beta) = \gamma \times \alpha + \gamma \times \beta, \tag{6}$$

so that a vector product is distributive.

To prove this we use the fact that the projection of any closed surface on any plane is zero if the sign of the projection of each portion is determined, as in § 76, by the cosine of the angle between the outward normal to the surface and a fixed normal to the plane. Apply this to the prism $OAB\text{-}CDE$ (Fig. 84), where OA is the vector α, AB the vector β, OB the vector $\alpha+\beta$, and OC the vector γ.

Then the vector $\frac{1}{2}(\alpha+\beta)\times\alpha$ is equal to the area of AOB, the vector $\frac{1}{2}\alpha\times(\alpha+\beta)$ is equal to the area of CDE, the vector $\beta\times\gamma$ is equal to the area $ABED$, the vector $\gamma\times(\alpha+\beta)$ is equal to the area $OBEC$, and the vector $\alpha\times\gamma$ is equal to the area $OADC$, and all these vectors are directed outward from the prism. Also the sum of the projections of these vectors on any line is zero, since it is equal to the sum of the projections of the faces of the prism on the plane perpendicular to that line. Hence the sum of these vectors is a vector whose projection on any line is zero, and therefore that vector is zero. Hence

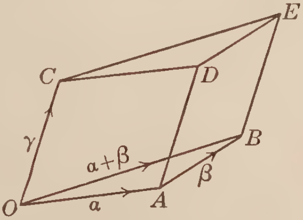

FIG. 84

$$\tfrac{1}{2}(\alpha+\beta)\times\alpha+\tfrac{1}{2}\alpha\times(\alpha+\beta)+\beta\times\gamma+\gamma\times(\alpha+\beta)+\alpha\times\gamma=0.$$

Reducing this by the aid of (4), we have

$$-\gamma\times\beta+\gamma\times(\alpha+\beta)-\gamma\times\alpha=0;$$

whence (6) follows.

From (6) follows

$$(\gamma+\delta)\times(\alpha+\beta)=\gamma\times\alpha+\gamma\times\beta+\delta\times\alpha+\delta\times\beta, \tag{7}$$

as in ordinary multiplication, with the single exception that the order of the factors must be carefully preserved. The extension to any number of vectors in each factor is obvious.

As in the case of the scalar product, so-called "cancellation" in a vector equation must be avoided. The equation

$$\alpha\times\gamma=\alpha\times\beta$$

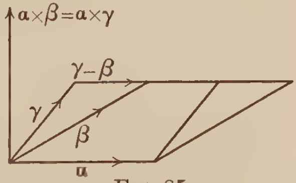

FIG. 85

leads to $\alpha\times(\gamma-\beta)=0$, which, by (3), means in general that α is parallel to $\gamma-\beta$. This is shown in Fig. 85, where the area of the parallelogram defined by α and β is equal to that of the parallelogram defined by α and γ.

Let i, j, k be as in § 81. Then

$$\begin{gathered} \mathbf{i}\times\mathbf{i}=0, \qquad \mathbf{j}\times\mathbf{j}=0, \qquad \mathbf{k}\times\mathbf{k}=0, \\ \mathbf{i}\times\mathbf{j}=-\mathbf{j}\times\mathbf{i}=\mathbf{k}, \quad \mathbf{j}\times\mathbf{k}=-\mathbf{k}\times\mathbf{j}=\mathbf{i}, \quad \mathbf{k}\times\mathbf{i}=-\mathbf{i}\times\mathbf{k}=\mathbf{j}. \end{gathered} \tag{8}$$

Then, if
$$\alpha = A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k},$$
$$\beta = A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k},$$

and we form $\alpha \times \beta$ by (7) and reduce by (8), we have

$$\begin{aligned} \alpha \times \beta &= (B_1C_2 - B_2C_1)\mathbf{i} + (C_1A_2 - C_2A_1)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix}. \end{aligned}$$

84. Curves. From the origin of coördinates O draw a vector $OP = \mathbf{r}$. Then, if $\mathbf{r}$ is a function both in direction and magnitude of a single parameter t, the extremity of $\mathbf{r}$ describes a curve, the equation of which may be written

$$\mathbf{r} = \mathbf{f}(t). \tag{1}$$

This may be brought into connection with the notation of § 51 by drawing the unit vectors i, j, k as in § 81. Then, if x, y, z are the coördinates of P,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \tag{2}$$

and if the equations of the curve are those given in § 51 equation (1) becomes

$$\mathbf{r} = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}; \tag{3}$$

so that (1) represents in one equation the three equations of § 51.

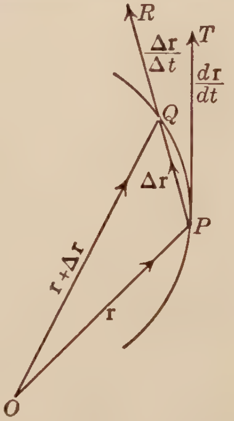

FIG. 86

Let P (Fig. 86) be the point corresponding to a certain value of t and let Q be the point corresponding to $t + \Delta t$. Then $OP = \mathbf{r}$, $OQ = \mathbf{r} + \Delta\mathbf{r}$, and, by the law of addition of vectors, $\Delta\mathbf{r} = PQ$. Hence $\frac{\Delta\mathbf{r}}{\Delta t}$ is the vector PR in the direction of the secant PQ. As Δt approaches zero as a limit, the point Q approaches P, and the secant approaches the tangent at P, under the assumption that the curve is continuous and has a tangent. At the same time, under the assumption that the curve has a length, the ratio of the chord PQ to the increment Δt is the same as the limit of the ratio of the arc PQ to Δt.

Hence we may write $\lim\limits_{t \to 0} \dfrac{\Delta \mathbf{r}}{\Delta t} = \dfrac{d\mathbf{r}}{dt},$

where $\dfrac{d\mathbf{r}}{dt}$ is a vector PT with the direction of the tangent to the curve and with a length $\dfrac{ds}{dt}$, s being the length of the curve.

Taking τ as a unit vector along the tangent, we have

$$\frac{d\mathbf{r}}{dt} = \tau \frac{ds}{dt}, \tag{4}$$

or, in differential notation,

$$d\mathbf{r} = \tau \, ds. \tag{5}$$

This result may be checked from (2) since

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}, \tag{6}$$

which is a vector with the direction of the tangent and with the length

$$\sqrt{dx^2 + dy^2 + dz^2} = ds.$$

Let **F** be any function which has both direction and magnitude at each point of a certain region. Such a function is a *vector function*, and we may write

$$\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

Then, if **F** makes an angle ϕ with the direction of the curve, and $|\mathbf{F}| = F$, we have $\mathbf{F} \cdot d\mathbf{r} = F \cos \phi \, ds$, which is the product of the projection of **F** on ds multiplied by ds. On the other hand,

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$$

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k};$$

whence

$$\mathbf{F} \cdot d\mathbf{r} = P\,dx + Q\,dy + R\,dz. \tag{7}$$

A simple illustration is obtained by letting **F** be a force. Then P, Q, R are the components of the force, and $\mathbf{F} \cdot d\mathbf{r} = F \cos \phi \, ds$ is the element of work done in moving through a length ds of the curve.

85. Areas. A vector is a magnitude with which a definite direction is associated. Now a plane area determines a definite direction; namely, that of the normal to the plane. Hence a plane area may be represented by a vector whose magnitude is equal to the scalar area and whose direction is perpendicular to the plane. Let there be given an area of scalar magnitude A lying in a plane the normal to which makes angles α, β, γ with OX, OY, OZ respectively. The vector **A** is then a vector of length A making angles α, β, γ with the coördinate axes. The projections of this vector on the axes are, respectively, $A\cos\alpha$, $A\cos\beta$, $A\cos\gamma$. Hence

$$\mathbf{A} = A\cos\alpha\,\mathbf{i} + A\cos\beta\,\mathbf{j} + A\cos\gamma\,\mathbf{k}. \tag{1}$$

It is to be noticed that $A\cos\alpha$ is the projection of A on the plane YOZ, that $A\cos\beta$ is the projection of A on the plane ZOX, and that $A\cos\gamma$ is the projection of A on the plane XOY.

If we have a surface S, there is obviously no definite direction connected with S. There is, however, a definite direction to each element dS; namely, that of the normal to the surface. We have, therefore, a vector element of area $d\mathbf{S}$ where, as in (1),

$$d\mathbf{S} = \cos\alpha\,dS\,\mathbf{i} + \cos\beta\,dS\,\mathbf{j} + \cos\gamma\,dS\,\mathbf{k}. \tag{2}$$

Now, if

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

is defined at all points of the surface S,

$$\mathbf{F}\cdot d\mathbf{S} = F\cos\phi\,dS = (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS, \tag{3}$$

where ϕ is the angle between $\mathbf{F}$ and the normal to S.

An example is obtained by letting $\mathbf{F} = \rho\mathbf{v}$, where ρ is the density of a fluid and $\mathbf{v}$ its velocity. Then $\mathbf{F}\cdot d\mathbf{S}$ is the amount of fluid per unit time which flows over the area dS (§ 77).

86. The gradient. We have defined vector functions in § 84. In distinction, a function $F(x, y, z)$ to which a magnitude but no direction is assigned at a point (x, y, z) is called a scalar function.

We have seen in § 35 that for such a function we may construct a family of surfaces

$$F(x, y, z) = c, \tag{1}$$

and that the maximum rate of change of F takes place in a direction normal to these surfaces and is equal to $\frac{dF}{dn}$, where n is measured along the normal.

Let ν be a vector of unit length and let us write

$$\nabla F = \frac{dF}{dn}\nu. \tag{2}$$

Then ∇F (read "del F") is a vector function which gives in direction and magnitude the maximum rate of change of the function F at each point of space for which F is defined. It is called the *gradient* of F.

We have seen in § 35 that if a distance s is measured in a direction making an angle ϕ with n, then

$$\frac{dF}{ds} = \frac{dF}{dn}\cos\phi. \tag{3}$$

We may apply (3) to distances measured parallel to the coördinate axes and obtain

$$\frac{\partial F}{\partial x}=\frac{dF}{dn}\cos\alpha,\quad \frac{\partial F}{\partial y}=\frac{dF}{dn}\cos\beta,\quad \frac{\partial F}{\partial z}=\frac{dF}{dn}\cos\gamma, \tag{4}$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal and where the change from $\frac{dF}{dx}$ to $\frac{\partial F}{\partial x}$ etc. is made to conform to usage. Hence we have in (4) the components of ∇F. Consequently

$$\nabla F=\frac{\partial F}{\partial x}\mathbf{i}+\frac{\partial F}{\partial y}\mathbf{j}+\frac{\partial F}{\partial z}\mathbf{k}, \tag{5}$$

$$|\nabla F|=\frac{dF}{dn}=\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2}. \tag{6}$$

Equation (5) may be written

$$\nabla F=\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)F, \tag{7}$$

and we define
$$\nabla=\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k} \tag{8}$$

as the *operator del*. The manner in which del operates on a scalar function is shown in (7) as interpreted by (5).

The operator ∇ has many properties similar to those of differentiation. Thus

$$\begin{aligned}\nabla(F+G)&=\nabla F+\nabla G,\\ \nabla(FG)&=F\nabla G+G\nabla F,\\ \nabla\left(\frac{F}{G}\right)&=\frac{G\nabla F-F\nabla G}{G^2}.\end{aligned} \tag{9}$$

87. The divergence. In § 86 the operator ∇ has been applied to a scalar function. That operator may, however, be applied to a vector function and in two ways: either by analogy to a scalar product or by analogy to a vector product. The first method gives us, by definition,

$$\begin{aligned}\nabla\cdot\mathbf{F}&=\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(P\mathbf{i}+Q\mathbf{j}+R\mathbf{k})\\ &=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}.\end{aligned}$$

This is called the divergence of **F** and is written div **F**. We have

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{1}$$

If we apply this result, together with (3), § 85, to Green's theorem in the form (7), § 78, replacing the element $dx\,dy\,dz$ by a general volume element dV, we have

$$\iiint\limits_{(T)} \text{div}\,\mathbf{F}\,dV = \iiint\limits_{(T)} \nabla\cdot\mathbf{F}\,dV = \iint\limits_{(S)} \mathbf{F}\cdot d\mathbf{S}. \tag{2}$$

The reason for the choice of the name *divergence* may be seen by interpreting **F** as equal to $\rho\mathbf{v}$, where ρ is the density of a fluid and **v** its velocity. Then each integral in (2) is the amount of fluid per unit time which flows out of a space region. Applied to an infinitesimal volume it appears that div **F** represents the amount of fluid per unit time which streams or diverges from a point.

88. The curl. It is also possible to combine the operator ∇ with a vector function by analogy to a vector product. The result is a vector called the curl of **F**. We have, by definition,

$$\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}. \tag{1}$$

If we apply this to Stokes's theorem, (8), § 80, and use also (7), § 84, we have

$$\int\limits_{(C)} \mathbf{F}\cdot d\mathbf{r} = \iint\limits_{(S)} \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint\limits_{(S)} \nabla\times\mathbf{F}\cdot d\mathbf{S}. \tag{2}$$

The reason for the use of the word *curl* is hard to give without extended treatment of the subject of fluid motion. The student may obtain some help by noticing that if **F** is the velocity of a liquid, then for velocity in what we have called irrotational motion, curl $\mathbf{F} = 0$, and for vortex motion, curl $\mathbf{F} \neq 0$.

It may be shown that if a spherical particle of fluid be considered, its motion in a time dt may be analyzed into a translation, a deformation, and a rotation about an instantaneous axis. The curl of the vector **v** can be shown to have the direction of this axis and a magnitude equal to twice the instantaneous angular velocity.

EXERCISES

1. If α, β, γ are vectors to the three vertices of a triangle, show that the vector to a point two thirds of the way from any vertex to the middle point of the opposite side is $\dfrac{\alpha+\beta+\gamma}{3}$.

2. Find the direction and the magnitude of each of the vectors

$$\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}\alpha \quad \text{and} \quad \frac{\alpha\times\beta}{\alpha\cdot\alpha}\times\alpha.$$

3. Show that $\alpha\cdot(\beta\times\gamma)$ is the volume of a parallelepiped three edges of which are the vectors α, β, γ meeting in a point. When will the volume so computed be positive? negative?

4. From Ex. 3 show that

$$\alpha\cdot(\beta\times\gamma)=\beta\cdot(\gamma\times\alpha)=\gamma\cdot(\alpha\times\beta).$$

5. From Ex. 3 show that

$$\alpha\cdot(\beta\times\gamma)=0$$

is the condition that α, β, γ lie in the same plane or are parallel to the same plane.

6. Show that $$\alpha\cdot(\alpha\times\beta)=0.$$

7. From Ex. 3 show that the volume of a tetrahedron the edges of which are $(0, 0, 0)$, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is

$$\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

8. Using the unit vectors **i**, **j**, **k**, prove that

$$(\alpha\times\beta)\times(\gamma\times\delta)=(\alpha.\gamma\times\delta)\beta-(\beta\cdot\gamma\times\delta)\alpha.$$

9. Using the unit vectors **i**, **j**, **k**, prove that

$$(\alpha\times\beta)\cdot(\gamma\times\delta)=(\alpha\cdot\gamma)(\beta\cdot\delta)-(\beta\cdot\gamma)(\alpha\cdot\delta).$$

10. Using the unit vectors **i**, **j**, **k**, prove that

$$\alpha\times(\beta\times\gamma)=(\alpha\cdot\gamma)\beta-(\alpha\cdot\beta)\gamma,$$
$$(\alpha\times\beta)\times\gamma=(\alpha\cdot\gamma)\beta-(\gamma\cdot\beta)\alpha.$$

11. Given a triangle with vector sides α, β, γ, prove that

$$\gamma\cdot\gamma=(\alpha-\beta)\cdot(\alpha-\beta),$$

and thence obtain $$c^2=a^2+b^2-2\,ab\cos\theta.$$

12. Prove by vectors that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

13. Find the gradient of xyz.

14. Find the gradient of $x^2+y^2+z^2$.

15. Find the gradient of $\log (x^2 + y^2 + z^2)$.

16. Prove formulas (9), § 86.

17. Find the divergence and the curl of the vector function

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

18. Find the divergence and the curl of the vector function

$$\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}, \quad \text{where} \quad r = \sqrt{x^2 + y^2 + z^2}.$$

19. Find the gradient and the curl of the vector function

$$(bz - cy)\mathbf{i} + (cx - az)\mathbf{j} + (ay - bx)\mathbf{k}.$$

20. If $d\mathbf{r}$ is defined as in § 84, show that

$$\nabla \mathbf{f} \cdot d\mathbf{r} = df,$$

where df is the differential change in f in the direction $d\mathbf{r}$.

21. Show that the derivative (or differential) of a vector of constant length is perpendicular to the vector.

22. Show that if $\mathbf{r}$ is a vector of unit length, then $d\mathbf{r}$ is a vector perpendicular to $\mathbf{r}$ and equal to the angle $d\theta$ between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$.

23. For a curve show that $\mathbf{C} = \dfrac{d^2\mathbf{r}}{ds^2}$

is a vector whose direction is that of the principal normal and whose magnitude is that of the curvature, and hence that

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{1}{R}\,\mathbf{c},$$

where R is the radius of curvature and $\mathbf{c}$ a unit vector along the principal normal.

24. If $\mathbf{n}$, $\mathbf{t}$, and $\mathbf{c}$ are unit vectors along the binormal, tangent, and principal normal respectively, show that

$$\mathbf{T} = \frac{d\mathbf{n}}{ds} = \frac{d}{ds}(\mathbf{t} \times \mathbf{c})$$

is a vector whose magnitude is the torsion.

25. If a body describes a curve $\mathbf{r} = \mathbf{f}(t)$, the velocity is defined as $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}\cdot$ Show that $\mathbf{v} = v\boldsymbol{\tau}$, where v is the speed $\dfrac{ds}{dt}$ and $\boldsymbol{\tau}$ is a unit tangent vector.

26. A vector force being defined as

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{r}}{dt^2},$$

show that the components of force parallel to OX, OY, and OZ are

$$m\frac{d^2x}{dt^2}, \quad m\frac{d^2y}{dt^2}, \quad m\frac{d^2z}{dt^2}.$$

27. Placing $\mathbf{v} = v\tau$, where v is the speed $\frac{ds}{dt}$ and τ a unit vector along the tangent, show from Ex. 25, and using Ex. 23, that

$$\mathbf{F} = m\frac{d^2s}{dt^2}\tau + \frac{mv^2}{R}\mathbf{n}.$$

Hence infer that the moving particle is acted on by two forces: one equal to $m\frac{d^2s}{dt^2}$ along the tangent, and the other equal to $\frac{mv^2}{R}$ directed toward the center of curvature.

28. Show that the vector area $d\mathbf{A}$ between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ is

$$d\mathbf{A} = \tfrac{1}{2}(\mathbf{r} \times d\mathbf{r}),$$

and hence that

$$\frac{d\mathbf{A}}{dt} = \tfrac{1}{2}(\mathbf{r} \times \mathbf{v}).$$

29. Show that if the force acting on a moving particle passes through a center O, then

$$\mathbf{r} \times \mathbf{F} = 0,$$

and hence the rate of charge of $\mathbf{A}$ is constant. Prove the converse.

30. If a curve is given in polar coördinates (r, θ), place

$$\mathbf{r} = r\mathbf{r}_1,$$

where r is the scalar length and $\mathbf{r}_1$ is a unit vector. Hence show that

$$\mathbf{v} = \frac{dr}{dt}\mathbf{r}_1 + r\frac{d\theta}{dt}\mathbf{n},$$

where $\mathbf{n}$ is a unit vector perpendicular to $\mathbf{r}$.

Then show that

$$\mathbf{F} = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{r}_1 + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{n},$$

and find the components of $\mathbf{F}$ along $\mathbf{r}$ and perpendicular to it.

31. A body is revolving about an axis OA with constant angular velocity ω. Let α be a vector in direction OA with magnitude equal to ω, $\mathbf{r}$ a vector from O to any point P of the vector, and $\mathbf{v}$ the velocity of P. Prove that

$$\mathbf{v} = \alpha \times \mathbf{r}.$$

32. Prove that if $\mathbf{v}$ is as in Ex. 31, curl $\mathbf{v} = 2\,\alpha$.

33. Prove that

$$\iint_{(S)} \mathbf{G}\nabla\cdot\mathbf{F}\, d\mathbf{S} = \iiint_{(T)} \nabla\mathbf{F}\cdot\nabla\mathbf{G}\, dV + \iiint_{(T)} \mathbf{G}\nabla\cdot\nabla\mathbf{F}\, dV.$$

CHAPTER X

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

89. Introduction. The equation

$$f(x, y, c) = 0, \tag{1}$$

where c is an arbitrary constant, defines a family of curves, one curve of the family being determined by a given value of c.

The direction of a curve of the family at any point is given by

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}, \tag{2}$$

which in general involves c.

By § 40, equation (1) determines c in general as a function of x and y, and the substitution of this value of c in (2) gives an equation of the form

$$F\left(x, y, \frac{dy}{dx}\right) = 0. \tag{3}$$

This gives a relation between a point and a direction of a curve through that point which is true for any point and any curve of the family (1). It is called the differential equation of the family.

Conversely, to any given equation of the form (3) corresponds an equation of form (1). This we shall prove in the following section, but it may be made graphically plausible as follows:

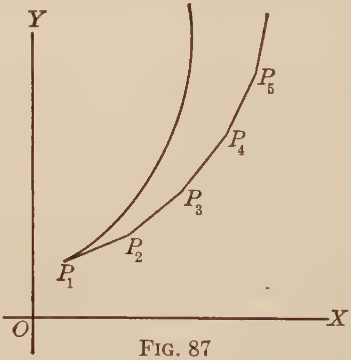

FIG. 87

If the coördinates of a point P_1 are assigned to x and y in (3), that equation determines one or more directions through P_1 (Fig. 87). Following one of these directions, we may determine another point, P_2. If the coördinates of P_2 are substituted in (3), a direction is determined by means of which a third point, P_3, is found. Proceeding in this way, we trace a broken line such that the coördinates of every vertex and the direction of the following segments satisfy (3).

Now it may be shown that this broken line approaches a curve as a limit as the length of each segment approaches zero, and this curve has the property that the coördinates of any point on it and its direction at that point satisfy (3).

Since in this construction P_1 may be any point of the plane, there is evidently a family of curves satisfying (3). The constant c in the equation (1) may be taken, for example, as the ordinate of the point in which a curve of the family cuts the axis of y or any other line $x = x_1$.

90. Existence proof. Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

with the assumption that $f(x, y)$ may be expanded into a power series in the neighborhood of $x = x_0$, $y = y_0$. Without loss of generality we may take $x_0 = 0$, $y_0 = 0$, since this amounts to a change of coördinates, and write (1) in the form

$$\frac{dy}{dx} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots. \tag{2}$$

In (2) substitute

$$y = c_1x + c_2x^2 + c_3x^3 + \cdots. \tag{3}$$

The coefficients c_i are readily obtained by comparing like powers of x. We have

$$\begin{aligned} c_1 &= a_{00}, \\ 2\,c_2 &= a_{10} + a_{01}c_1, \\ 3\,c_3 &= a_{20} + a_{11}c_1 + a_{02}{c_1}^2 + a_{01}c_2, \\ 4\,c_4 &= a_{30} + a_{21}c_1 + a_{12}{c_1}^2 + a_{03}{c_1}^3 + 2\,a_{02}c_1c_2 + a_{11}c_2 + a_{01}c_3, \end{aligned}$$

etc.

The coefficients of (3) are then completely determined. If (3) converges, it defines a function y which satisfies equation (1). We must therefore prove the convergence of (3). For that purpose consider the equation

$$\frac{dy}{dx} = \frac{M}{\left(1 - \frac{x}{r}\right)\left(1 - \frac{y}{\rho}\right)} = b_{00} + b_{10}x + b_{01}y + \cdots, \tag{4}$$

where M, r, ρ are determined as in § 28 with reference to the series (2). The function

$$\frac{M}{\left(1-\frac{x}{r}\right)\left(1-\frac{y}{\rho}\right)}$$

dominates the function $f(x, y)$, so that $b_{ik} > |a_{ik}|$. Then, if we solve (4) by a series

$$y = C_1x + C_2x^2 + C_3x^3 + \cdots, \tag{5}$$

the coefficients $C_i > |c_i|$.

Hence if (5) converges, so does (3).

But equation (4) may be solved directly, since it may be written

$$\left(1-\frac{y}{\rho}\right)dy = \frac{M}{1-\frac{x}{r}}\,dx;$$

whence

$$y - \frac{y^2}{2\,\rho} = -\,Mr \log\left(1-\frac{x}{r}\right),$$

the constant of integration being so taken that $y = 0$ when $x = 0$. Then

$$y = \rho - \sqrt{\rho^2 + 2\,\rho Mr \log\left(1-\frac{x}{r}\right)}, \tag{6}$$

where we take the sign of the radical so that $y = 0$ when $x = 0$. Now by direct application of Maclaurin's expansion of a function, (6) may be expanded into a convergent series which can be no other than (5). Hence (3) converges.

We have shown that for any point (x_0, y_0) for which $f(x, y)$ has a series expansion, there is one and only one solution of (1). If $f(x, y)$ is a multiple-valued function, there will be a series expansion and a solution corresponding to each value of the function. For example, for

$$\frac{dy}{dx} = x \pm \sqrt{x^2 + y^2} \tag{7}$$

there will be through each point two solutions corresponding to the two signs of the radical.

Also if $f(x, y)$ cannot be expanded into a series, the proof fails. This may happen at a point for which $f(x, y)$ becomes infinite or indeterminate. For example, consider

$$\frac{dy}{dx} = \frac{y}{x}. \tag{8}$$

Our method fails for any point for which $x = 0,\ y \neq 0$. But this difficulty may be removed by writing the equation as

$$\frac{dx}{dy} = \frac{x}{y}$$

and finding x as a series expansion in y. A more fundamental difficulty occurs when $x = 0,\ y = 0$, for then the right-hand side of (8) is indeterminate. In fact, the solution of (8) is

$$y = cx,$$

and through the origin go all the lines of the solution.

91. Equations of the first degree. The problem of proceeding from a differential equation (3), § 89, to its solution (1), § 89, is a difficult one which can be solved explicitly only in the simpler cases. We shall consider in this section equations in which $\frac{dy}{dx}$ appears in the first power only, so that the equation is of the form

$$M\,dx + N\,dy = 0. \tag{1}$$

We have the following cases:

CASE I. *Variables separable.* If the equation is in the form

$$f_1(x)dx + f_2(y)dy = 0, \tag{2}$$

the variables are said to be separated. The solution is then

$$\int f_1(x)dx + \int f_2(y)dy = c, \tag{3}$$

where c is an arbitrary constant.

The variables can be separated if M and N can each be factored into two factors one of which is a function of x alone and the other of y alone. The equation may then be divided by the factor of M which contains y multiplied by the factor of N which contains x.

For example, $(1 - x^2)dy + (xy - ax)dx = 0$

may be written $\dfrac{dy}{y - a} + \dfrac{x\,dx}{1 - x^2} = 0,$

which gives $\log(y - a) - \frac{1}{2}\log(1 - x^2) = c,$

or $\log \dfrac{y - a}{\sqrt{1 - x}} = c;$

whence follows $y - a = k\sqrt{1 - x^2},$

where k is an arbitrary constant.

CASE II. *Homogeneous equation.* When M and N are homogeneous functions (§ 34) of the same degree n, equation (1) is said to be homogeneous and may be solved as follows:

Place $y = vx$. Then $dy = v\,dx + x\,dv$, and (1) takes the form

$$x^n f_1(v)dx + x^n f_2(v)(v\,dx + x\,dv) = 0, \tag{4}$$

and the variables are easily separated. The substitution $x = vy$ may also be made if more convenient.

For example, in $(x^2 - y^2)dx + 2\,xy\,dy = 0$

place $y = vx$. There results

$$\frac{dx}{x} + \frac{2\,v\,dv}{1 + v^2} = 0;$$

whence $$x(1 + v^2) = c.$$

Replacing v by $\frac{y}{x}$ gives the solution

$$x^2 + y^2 = cx.$$

CASE III. *Equation with linear coefficients.* The equation

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0 \tag{5}$$

may usually be made homogeneous as follows:

Place $$x = x' + h, \qquad y = y' + k,$$

after determining h and k so as to satisfy the two equations

$$a_1h + b_1k + c_1 = 0 \quad \text{and} \quad a_2h + b_2k + c_2 = 0. \tag{6}$$

The differential equation then becomes

$$(a_1x' + b_1y')dx' + (a_2x' + b_2y')dy' = 0, \tag{7}$$

which is homogeneous.

An exception to this method occurs when equations (6) cannot be solved for h and k. In this case $a_2 = ka_1$, $b_2 = kb_1$, where k is some constant. Then, by placing $a_1x + b_1y = x'$ the variables x and x' are easily separated and the equations can be solved.

CASE IV. *Linear equation.* The equation

$$\frac{dy}{dx} + Py = Q, \tag{8}$$

where P and Q are functions of x only, or are constants, is a linear equation of the first order.

Since $$\frac{d}{dx}\left(ye^{\int P\,dx}\right) = e^{\int P\,dx}\frac{dy}{dx} + P\,ye^{\int P\,dx},$$

equation (8) may be written as

$$\frac{d}{dx}\left(ye^{\int P\,dx}\right) = Qe^{\int P\,dx}; \qquad (9)$$

easier [illegible]

whence $$ye^{\int P\,dx} = \int Qe^{\int P\,dx}dx + c,$$

and $$y = e^{-\int P\,dx}\int Qe^{\int P\,dx}dx + ce^{-\int P\,dx}. \qquad (10)$$

For example, $$(1 - x^2)\frac{dy}{dx} + xy = ax$$

may be written $$\frac{dy}{dx} + \frac{x}{1 - x^2}y = \frac{ax}{1 - x^2},$$

which is of type (8) with $P = \frac{x}{1 - x^2}$, $Q = \frac{ax}{1 - x^2}$. Then

$$e^{\int P\,dx} = e^{\int \frac{x\,dx}{1-x^2}} = e^{-\frac{1}{2}\log(1-x^2)} = \frac{1}{\sqrt{1 - x^2}}.$$

Therefore $$y = \sqrt{1 - x^2}\int\frac{ax}{(1 - x^2)^{\frac{3}{2}}}dx + c\sqrt{1 - x^2}$$
$$= a + c\sqrt{1 - x^2}.$$

CASE V. *Bernoulli's equation.* The equation

$$\frac{dy}{dx} + Py = Qy^n, \qquad (11)$$

where P and Q are functions of x, or constants, is a Bernoulli equation. It may be made linear by dividing by y^n and substituting $y^{1-n} = z$.

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CASE VI. *Exact equation.* By § 36, when

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

equation (1) may be written $df = 0$, (12)

the solution of which is $f = c$.

The function f may be found either by the method of § 36 or by that of § 75.

CASE VII. *Solution by integrating factors.* The equation

$$M\,dx + N\,dy = 0 \tag{13}$$

always has a solution of the form

$$f(x, y, c) = 0,$$

which, by the theory of implicit functions, may be written

$$\phi(x, y) = c. \tag{14}$$

From (14) we get

$$\frac{dy}{dx} = -\frac{\dfrac{\partial\phi}{\partial x}}{\dfrac{\partial\phi}{\partial y}},$$

which must agree with (13). Hence there exists a function μ such that

$$\frac{\partial\phi}{\partial x} = \mu M$$

and

$$\frac{\partial\phi}{\partial y} = \mu N. \tag{15}$$

Then

$$\mu(M\,dx + N\,dy) = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy = d\phi. \tag{16}$$

The function μ is called an *integrating factor*. Our work shows that an integrating factor always exists, and that if an equation is multiplied by it the equation becomes exact.

There are an infinite number of integrating factors for a given equation. For if ϕ is determined by (16), and $f(\phi)$ is any function of ϕ, then

$$\mu f(\phi)(M\,dx + N\,dy) = f(\phi)d\phi = dF, \tag{17}$$

so that $\mu f(\phi)$ is an integrating factor.

No general method is known for finding integrating factors, but the factors are known for certain cases. We give a list of the simpler cases, leaving it as an exercise for the student to verify by differentiation that each of the equations mentioned satisfies the condition for an exact differential equation after it is multiplied by the proper factor.

1. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$, then $e^{\int f(x)dx}$ is an integrating factor.

2. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = f(y)$, then $e^{-\int f(y)dy}$ is an integrating factor.

3. If M and N are homogeneous and of the same degree, then $\dfrac{1}{xM + yN}$ is an integrating factor.

4. If $M = yf_1(xy)$ and $N = xf_2(xy)$, then $\dfrac{1}{xM - yN}$ is an integrating factor.

5. $e^{\int f_1(x)dx}$ is an integrating factor of the linear equation

$$\frac{dy}{dx} + yf_1(x) = f_2(x).$$

As a practical point the student should look for an integrating factor only after he has tried to integrate by other methods.

As an example, consider

$$(4\,x^2y - 3\,y^2)dx + (x^3 - 3\,xy)dy = 0.$$

Here

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{1}{x}.$$

Consequently $e^{\int \frac{1}{x}dx} = x$ is an integrating factor. After multiplication by the factor the equation becomes

$$(4\,x^3y - 3\,xy^2)dx + (x^4 - 3\,x^2y)dy = 0,$$

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$\phi(x, y) = \int P(x, y_0)dx$

$+\int Q($ [illegible]

the integral of which is $x^4y - \frac{3}{2}\,x^2y^2 = c.$

CASE VIII. *Solution by series.* Let the differential equation be put into the form

$$\frac{dy}{dx} = f(x, y). \tag{18}$$

Cons

We may then compute

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx},$$

$$\frac{d^3y}{dx^3} = \frac{\partial^2 f}{\partial x^2} + 2\,\frac{\partial^2 f}{\partial x\,\partial y}\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dx^2},$$

and all succeeding derivatives. We may then substitute the values $x = x_0$, $y = y_0$, obtaining $\left(\dfrac{dy}{dx}\right)_0$, $\left(\dfrac{d^2y}{dx^2}\right)_0$, etc., and the Taylor series

$$y = y_0 + \left(\frac{dy}{dx}\right)_0(x - x_0) + \frac{1}{2\,!}\left(\frac{d^2y}{dx^2}\right)_0(x - x_0)^2 + \cdots, \tag{19}$$

which is a solution of the differential equation.

Another way is to substitute in (18) the series

$$y = y_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \tag{20}$$

and determine a_1, a_2, $\cdots$ by the method of undetermined coefficients. If $f(x, y)$ is expanded into a Taylor series in the neighborhood of (x_0, y_0), equation (18) takes the form

$$\begin{aligned}\frac{dy}{dx} = c_0 + c_1(x - x_0) + c_2(y - y_0) + c_3(x - x_0)^2 \\ + c_4(x - x_0)(y - y_0) + c_5(y - y_0)^2 + \cdots.\end{aligned}$$

Substituting from (20), we have

$$\begin{aligned}a_1 + 2\,a_2(x - x_0) + 3\,a_3(x - x_0)^2 + \cdots = c_0 + c_1(x - x_0) \\ + c_2[a_1(x - x_0) + a_2(x - x_0)^2 + \cdots] \\ + c_3(x - x_0)^2 + c_4(x - x_0)[a_1(x - x_0) + \cdots] \\ + c_5[a_1(x - x_0) + \cdots]^2 + \cdots;\end{aligned}$$

whence by equating coefficients of like powers of $(x - x_0)$,

$$\begin{aligned}a_1 &= c_0, \\ 2\,a_2 &= c_1 + c_2 a_1, \\ 3\,a_3 &= c_2 a_2 + c_3 + c_4 a_1 + c_5 a_1^2, \\ &\text{etc.}\end{aligned}$$

As an example, consider

$$\frac{dy}{dx} = x^2 + y^2,$$

and let us look for a series in ascending powers of x. We have $x_0 = 0$, $y_0 = c$, c being arbitrary.

By the first method,

$= f'x_0 = f'(a)$

$$\frac{dy}{dx} = x^2 + y^2, \quad \left(\frac{dy}{dx}\right)_0 = c^2,$$

$$\frac{d^2y}{dx^2} = 2\,x + 2\,y\,\frac{dy}{dx}, \quad \left(\frac{d^2y}{dx^2}\right)_0 = 2\,c^3,$$

$$\frac{d^3y}{dx^3} = 2 + 2\left(\frac{dy}{dx}\right)^2 + 2\,y\,\frac{d^2y}{dx^2}, \quad \left(\frac{d^3y}{dx^3}\right)_0 = 2 + 6\,c^4,$$

$$\frac{d^4y}{dx^4} = 6\,\frac{d^2y}{dx^2}\,\frac{dy}{dx} + 2\,y\,\frac{d^3y}{dx^3}, \quad \left(\frac{d^4y}{dx^4}\right)_0 = 4\,c + 24\,c^5.$$

Therefore

$$y = c + c^2x + c^3x^2 + \frac{1 + 3c^4}{3}x^3 + \frac{c + 6c^5}{6}x^4 + \cdots.$$

By the second method, place

$$y = c + a_1x + a_2x^2 + a_3x^3 + \cdots$$

and substitute in the equation. We have

$$\begin{aligned} &a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots \\ &\quad = x^2 + (c + a_1x + a_2x^2 + a_3x^3 + \cdots)^2 \\ &\quad = c^2 + 2a_1cx + (1 + a_1{}^2 + 2a_2c)x^2 + (2a_1a_2 + 2a_3c)x^3 + \cdots; \end{aligned}$$

whence $a_1 = c^2,\ a_2 = c^3,\ a_3 = \frac{1 + 3c^4}{3},\ a_4 = \frac{c + 6c^5}{6},$

as before.

92. Equations not of the first degree. We may write equation (3), § 89, in the form

$$F(x, y, p) = 0, \tag{1}$$

where $p = \frac{dy}{dx}$. This is the general differential equation of the first order and may sometimes be solved in the following cases:

CASE I. *Equations solvable for p.* If (1) is considered as an equation for p, it may sometimes be solved into a number of distinct equations of the type

$$p = \phi(x, y). \tag{2}$$

If $f(x, y, c) = 0$ is a solution of (2), it is obviously a solution of (1). We shall have as many solutions of (1) as we have equations (2), and these solutions may be left distinct or combined into one by multiplication.

As an example, consider

$$x^2p^3 - 2p^2x^2 + (2x^2y - x^4 - y^2)p - 2(2x^2y - x^4 - y^2) = 0.$$

This may be solved into the three equations

$$p = 2, \quad p = \frac{y}{x} - x, \quad p = -\frac{y}{x} + x,$$

the solutions of which are

$$p = 2x + c, \quad y + x^2 - cx = 0, \quad x^3 - 3xy + c = 0,$$

and the solution of the original equation may be written

$$(y - 2x + c)(y + x^2 - cx)(x^3 - 3xy + c) = 0.$$

As another example, the equation

$$y\left(\frac{dy}{dx}\right)^2+2\,x\frac{dy}{dx}-y=0$$

gives

$$p=-\frac{x}{y}\pm\frac{\sqrt{x^2+y^2}}{y},$$

the solution of which is $y^2+2\,cx-c^2=0$,

independently of the sign of the radical.

CASE II. *Equations solvable for y.* Solving (1) for y, we may have one or more equations of the form

$$y=f(x,\,p). \tag{3}$$

Differentiating with respect to x and replacing $\frac{dy}{dx}$ by p, we have an equation of the form

$$p=\phi\left(x,\,p,\,\frac{dp}{dx}\right), \tag{4}$$

where p and x are the variables.

Let the solution of (4) be of the form

$$\psi(x,\,p,\,c)=0. \tag{5}$$

The elimination of p from (3) and (5) gives an equation between x, y, and c which is in general the solution of (1). But the process of elimination may bring in extraneous factors, and the solution should be tested by substitution in (1).

If the elimination cannot be performed, equations (3) and (5) may be taken simultaneously as the parametric form of the solution, with p as the parameter.

As an example, the equation

$$xp^2-2\,yp+ax=0 \tag{6}$$

may be written

$$y=\frac{xp}{2}+\frac{ax}{2\,p};$$

whence, by differentiating,

$$p=\frac{1}{2}\left(p+\frac{a}{p}\right)+\left(\frac{x}{2}-\frac{ax}{2\,p^2}\right)\frac{dp}{dx},$$

or

$$\left(\frac{a}{p}-p\right)\left(1-\frac{x}{p}\frac{dp}{dx}\right)=0. \tag{7}$$

The first factor gives $p=\pm\sqrt{a}$, and this value, substituted in (6), gives

$$y=\pm\,x\sqrt{a}. \tag{8}$$

This is found on trial to satisfy (6).

The second factor of (7) gives

$$\frac{dp}{p}-\frac{dx}{x}=0;$$

whence $p=cx$. If this is substituted in (6) it gives

$$y=\frac{cx^2}{2}+\frac{a}{2c}, \tag{9}$$

which also satisfies (6).

It appears that (9) contains an arbitrary constant and therefore represents the family of curves which form the general solution. The solution (8), however, is peculiar, or *singular*, in that it contains no arbitrary constant and does not belong to the family (9). Singular solutions will be discussed in § 95.

A note of caution is necessary here. The student may be tempted to integrate each of the equations

$$p=\pm\sqrt{a} \quad \text{and} \quad p=cx$$

instead of substituting in (6). If he does, he will find

$$y=\pm x\sqrt{a}+c_1, \quad y=\frac{cx^2}{2}+c_2. \tag{10}$$

However, the constants of integration are not arbitrary, for substitution in (6) gives $c_1=0$, $c_2=\frac{a}{2c}\cdot$ Hence (10) agrees with (8) and (9).

Case III. *Clairaut's equation.* The equation

$$y=px+f(p) \tag{11}$$

is called Clairaut's equation. It is a special but important case of an equation solved for y.

Differentiating with respect to x, we have

$$[x+f'(p)]\frac{dp}{dx}=0. \tag{12}$$

The factor $\frac{dp}{dx}=0$ gives $p=c$; whence the general solution of the equation is

$$y=cx+f(c), \tag{13}$$

which may obviously be written down at sight of the equation. Equation (13) represents a family of straight lines. The other factor in (12) combined with the given equation gives

$$\begin{aligned} x&=-f'(p),\\ y&=-pf'(p)+f(p), \end{aligned} \tag{14}$$

a parametric form of another solution of the equation. This is a singular solution. It is a curve for which the direction at any point is p. Hence the equation of the tangent line at a point for which $p = c$ is

$$y + cf'(c) - f(c) = c[x + f'(c)],$$

which reduces to

$$y = cx + f(c).$$

Hence the lines (13) are the tangent lines to the curve (14).

For example, consider

$$y = px + a\sqrt{1 + p^2}. \tag{15}$$

The general solution is

$$y = cx + a\sqrt{1 + c^2}, \tag{16}$$

where c is arbitrary. The second solution is

$$x = -\frac{ap}{\sqrt{1 + p^2}},$$

$$y = \frac{a}{\sqrt{1 + p^2}}.$$

Eliminating p, we have

$$x^2 + y^2 = a^2, \tag{17}$$

a circle to which the lines (16) are tangent.

CASE IV. *Equations solvable for x.* Solving (1) for x, we have one or more equations of the form

$$x = \phi(y, p). \tag{18}$$

Differentiating with respect to y and placing $\frac{dx}{dy} = \frac{1}{p}$, we have

$$\frac{1}{p} = \psi\left(y, p, \frac{dp}{dy}\right). \tag{19}$$

If this can be solved for p, the elimination of p between (1) and (19) gives the solution of (1).

As an example, consider

$$x - 2p - \log p = 0. \tag{20}$$

Solving for x and differentiating with respect to y, we have

$$\frac{1}{p} = \left(2 + \frac{1}{p}\right)\frac{dy}{dp};$$

whence

$$dy = (2p + 1)dp,$$

and

$$y = p^2 + p + c. \tag{21}$$

Since the result of eliminating p from (20) and (21) is complicated, we take

$$\begin{aligned} x &= 2p + \log p, \\ y &= p^2 + p + c, \end{aligned} \tag{22}$$

as the parametric form of the equation (20). In (22) p may be given any value, and x and y may then be found. In this way the curve may be sketched.

93. Envelope of a family of plane curves. In the previous section we have met examples of a family of curves each of which is tangent to the same curve. When that happens, the latter curve is said to be the envelope of the family. Obviously, any curve is the envelope of its tangent lines. Let

$$f(x, y, c) = 0 \tag{1}$$

be a family of curves (Fig. 88), and let C be a curve which is tangent to each curve of the family and such that each point of C is a point of tangency of some curve of (1). Then each point is determined by c of (1), and therefore if $P(x, y)$ is such a point we have

$$x = \phi_1(c), \qquad y = \phi_2(c), \tag{2}$$

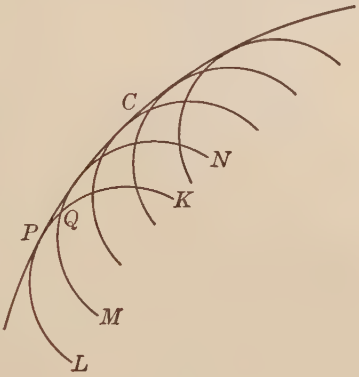

FIG. 88

which is the parametric equation of the curve C. But the x and y of (2) satisfy (1). Hence we have

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial c}\,dc = 0. \tag{3}$$

In this equation x and y are determined by (2), and therefore $\frac{dy}{dx}$ is the slope of C. But the slope of (1), with c constant, is given by

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0; \tag{4}$$

and since C and (1) are tangent their slopes are the same, and we have, by comparison of (3) and (4),

$$\frac{\partial f}{\partial c} = 0. \tag{5}$$

Equations (1) and (5) together will determine x and y as functions of c, as in (2). The elimination of c between these equations will give the equation of C in x and y.

Therefore we say, *The envelope of a family of curves* (1), *if such exists, is found by eliminating c from equations* (1) *and* (5).

For example, consider

$$(x-2c)^2+y^2=c^2,$$

a family of circles as shown in Fig. 89. The envelope is found by eliminating c from this equation and

$$-4(x-2c)=2c.$$

The result is

$$y=\pm\frac{x}{\sqrt{3}},$$

two straight lines to which each circle is tangent. The equation of the envelope may also be written in the parametric form (2) as

$$x=\frac{3c}{2},\quad y=\pm\frac{\sqrt{3}}{2}c.$$

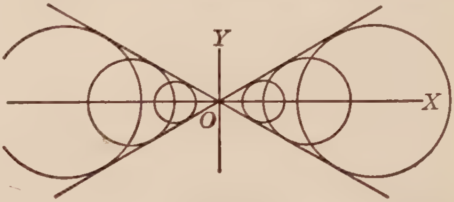

FIG. 89

It does not follow from what has been said that a family of curves necessarily has an envelope, nor that the elimination of c between (1) and (5) may not give curves which are not envelopes.

For example, if we apply the method to the family of parabolas (Fig. 90)

$$y^2=cx,$$

the equation $\frac{\partial f}{\partial c}=0$ is $x=0$, and the elimination of c gives the point $x=0$, $y=0$ and not a curve. The family has no envelope except that in a certain generalization the point O may be called an envelope.

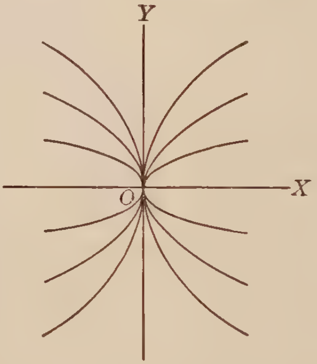

FIG. 90

Again, consider the family (Fig. 91)

$$(y-c)^2=x(x-1).$$

Here $\frac{\partial f}{\partial c}=0$ is $y-c=0$, and the elimination of c gives

$$x=0\quad\text{and}\quad x=1.$$

A glance at the figure shows that $x=0$ is an envelope, but the line $x=1$ is the locus of the double points and is not an envelope.

In fact, if there is a *singular* point for which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ on each curve of the family, the locus of that point will always appear as part of the curve found by eliminating c between (1) and (5). For if there is a singular point on each curve, the locus of the singular points is

$$x = \psi_1(c), \qquad y = \psi_2(c).$$

Then, from (1),

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial c}\,dc = 0,$$

which reduces to $\frac{\partial f}{\partial c} = 0,$

so that the singular points satisfy equations (1) and (5) simultaneously.

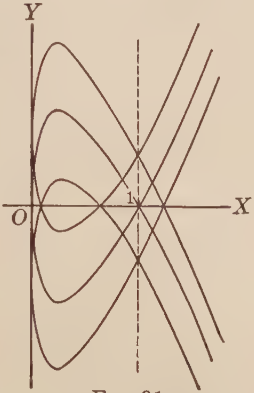

FIG. 91

Other extraneous factors may appear in handling equations (1) and (5). Hence it is necessary to test geometrically a solution found for an envelope to see if it really is an envelope.

We shall prove in the next section the theorem that the envelope is the limit of points of intersection of two neighboring curves of the family.

94. Envelope as locus of limit points. Let

$$f(x, y, c) = 0 \tag{1}$$

be a family of plane curves, and let LK (Fig. 88, § 93) be a particular curve of the family corresponding to a definite value of c. Let c be given an increment Δc. Then LK is displaced to a position MN, the equation of which is

$$f(x, y, c + \Delta c) = 0. \tag{2}$$

The two curves LK and MN intersect in a point Q, the coördinates of which are found by solving equations (1) and (2) or, what is the same thing, by solving equation (1) and

$$\frac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c} = 0. \tag{3}$$

As $\Delta c \rightarrow 0$, the curve MN approaches coincidence with the curve LK, but the point Q will in general approach a definite limiting point P on the curve LK. We may call this point the *limit point* on the curve LK, and it may be found by solving the two equations

$$f(x, y, c) = 0,$$
$$\frac{\partial f}{\partial c} = 0, \tag{4}$$

where the last equation is obtained by taking the limit of the left-hand member of (3) as $\Delta c \rightarrow 0$. The locus of the limit points is the curve found by eliminating c from equations (4). This is the same locus found in § 93.

95. Singular solutions. The general solution of

$$f(x, y, p) = 0 \tag{1}$$

is a family of curves

$$F(x, y, c) = 0. \tag{2}$$

Any curve of the family (2) is such that the coördinates of any point on it and the slope of the curve at that point satisfy (1). Hence if the family (2) has an envelope, the equation of that envelope is also a solution of (1), since the slope of the envelope at any point is the same as the slope of some curve of (2) at the same point. The equation of the envelope is called the *singular solution* of (1), since it is not obtained by giving c a special value in (2). The first method of finding the singular solution is, then, to solve for (2) and then find the envelope of (2).

It is sometimes possible, however, to find the singular solution of (1) without first finding (2). A glance at Fig. 88 shows that at Q there are two values of p satisfying (1), but at P these two values coincide. Hence the singular solution of (1) is the locus of points for which two or more values of p in equation (1) coincide.

Now it is a well-known theorem of algebra that any multiple root of the equation

$$f(x) = 0$$

is also a root of

$$f'(x) = 0.$$

To prove this, note that if a is a multiple root of $f(x) = 0$, then

$$f(x) = (x - a)^r \phi(x);$$

whence

$$f'(x) = r(x - a)^{r-1}\phi(x) + (x - a)^r \phi'(x),$$

and the theorem is obvious by inspection.

Applying this to

$$f(x, y, p) = 0, \tag{3}$$

we see that a double root of this equation is also a root of

$$\frac{\partial f}{\partial p} = 0. \tag{4}$$

If p is eliminated from these two equations, the result is the locus of the points at which two curves of the family defined by the differential equation coincide, and this is in general the singular solution of (1).

It should be noticed that either method of finding the singular solution may lead to extraneous solutions, and any apparent solution should be tested by substitution in (1).

For example, consider the Clairaut equation

$$y = px + a\sqrt{1 + p^2}. \tag{5}$$

The general solution is the family of straight lines

$$y = cx + a\sqrt{1 + c^2}, \tag{6}$$

the envelope of which is the circle

$$x^2 + y^2 = a^2. \tag{7}$$

On the other hand, (5) may be written as the quadratic equation

$$(x^2 - a^2)p^2 - 2\,xyp + (y^2 - a^2) = 0,$$

which gives two equal values of p when

$$x^2y^2 - (x^2 - a^2)(y^2 - a^2) = 0,$$

which reduces to (7). By trial (7) is seen to satisfy (5) and is therefore the singular solution of (5).

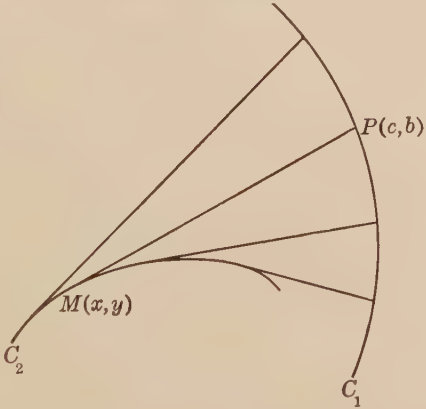

FIG. 92

96. Evolute and involute. The evolute of a curve is the envelope of its normals. Let the equation of a curve C_1 be

$$y = f(x), \tag{1}$$

and let $P(c, b)$ (Fig. 92) be a point on it. Then

$$b = f(c).$$

The equation of the normal at (c, b) is

$$y - f(c) = -\frac{1}{f'(c)}(x - c), \tag{2}$$

from a well-known formula of analytic geometry.

This is a family of straight lines involving one parameter c. To find the envelope we differentiate equation (2) with respect to c, obtaining

$$-f'(c)=\frac{f''(c)}{[f'(c)]^2}(x-c)+\frac{1}{f'(c)}, \tag{3}$$

and equations (2) and (3) give

$$\begin{aligned} x&=c-\frac{1+[f'(c)]^2}{f''(c)}f'(c),\\ y&=f(c)+\frac{1+[f'(c)]^2}{f''(c)}, \end{aligned} \tag{4}$$

which are the parametric equations of the evolute C_2. Here (x, y) are the coördinates of the point M corresponding to the point P.

From (4), $$(x-c)^2+[x-f(c)]^2=\frac{\{1+[f'(c)]^2\}^3}{[f''(c)]^2}. \tag{5}$$

The expression on the left of (5) is the square of the line MP, and the expression on the right is the square of the radius of curvature ρ of the curve (1) at the point (c, b). Hence the point M is the center of curvature of the point P, and therefore *the evolute of a curve is the locus of the centers of curvature of the given curve.*

We may write equations (4) in the form

$$x=c-\frac{\rho f'(c)}{\{1+[f'(c)]^2\}^{\frac{1}{2}}},\quad y=f(c)+\frac{\rho}{\{1+[f'(c)]^2\}^{\frac{1}{2}}}; \tag{6}$$

whence $$\begin{aligned} dx&=\left[1-\rho\frac{f''(c)}{\{1+[f'(c)]^2\}^{\frac{3}{2}}}\right]dc-\frac{f'(c)}{\{1+[f'(c)]^2\}^{\frac{1}{2}}}d\rho\\ &=-\frac{f'(c)}{\{1+[f'(c)]^2\}^{\frac{1}{2}}}d\rho, \end{aligned}$$

and $$\begin{aligned} dy&=\left[f'(c)-\frac{\rho f'(c)f''(c)}{\{1+[f'(c)]^2\}^{\frac{3}{2}}}\right]dc+\frac{1}{\{1+[f'(c)]^2\}^{\frac{1}{2}}}d\rho\\ &=\frac{1}{\{1+[f'(c)]^2\}^{\frac{1}{2}}}d\rho, \end{aligned}$$

and therefore $$dx^2+dy^2=d\rho^2;$$

whence, if s is the length of the curve C_2,

$$ds=d\rho,\qquad \rho=s+c, \tag{7}$$

or the length of the line MP is equal to the length of the curve C_2 measured from a proper origin, and therefore the curve C_1 may be unwound from C_2.

We have started with a given curve C_1 and have obtained C_2. Conversely, let us start with the curve C_2, lay off on the tangent lines to C_2 distances equal to s, where s is the length of C_2 from a fixed point, and find the locus of P. This locus is called the *involute.*

The equation of the tangent line at $M(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x);$$

and since this passes through $P(c, b)$,

$$\frac{dy}{dx} = \frac{y - b}{x - c}. \tag{8}$$

Also, by hypothesis,

$$(x - c)^2 + (y - b)^2 = s^2. \tag{9}$$

Let M move on the curve C_2. Then (x, y) and (c, b) both vary. Hence, from (9),

$$(x - c)dx + (y - b)dy - (x - c)dc - (y - b)db = s\,ds. \tag{10}$$

But

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= \frac{(y - b)^2 + (x - c)^2}{(x - c)^2}\,dx^2; \end{aligned}$$

whence

$$ds = \frac{s}{x - c}\,dx.$$

Also, from (8),

$$(x - c)dx + (y - b)dy = \frac{(x - c)^2 + (y - b)^2}{x - c}\,dx = \frac{s^2}{x - c}\,dx.$$

Hence equation (10) reduces to

$$(x - c)dc + (y - b)db = 0;$$

whence, from (8),

$$dc\,dx + db\,dy = 0,$$

that is, the tangents to C_2 are the normals of C_1. Then C_2 is the evolute of C_1. Since the point from which we measure s in C_2 is arbitrary, it follows that a curve has an infinite number of involutes but only one evolute.

97. Orthogonal trajectories of plane curves. A curve which intersects each curve of the family

$$f(x, y, c) = 0 \tag{1}$$

at a given angle is called a *trajectory*. In particular, if the given angle is a right angle the curve is an *orthogonal trajectory*.

To find the orthogonal trajectories of (1) we must first find the differential equation of the family (1). This is done by differentiating (1) with respect to x and eliminating c between the result and (1). We then have

$$F(x, y, p) = 0. \tag{2}$$

Now since the trajectories intersect (1) at right angles, the p of the trajectories is equal to minus the reciprocal of the p in equation (2). Hence if we replace p in (2) by $-\frac{1}{p}$, obtaining

$$F\left(x, y, -\frac{1}{p}\right) = 0, \tag{3}$$

we have the differential equation of the orthogonal trajectories.

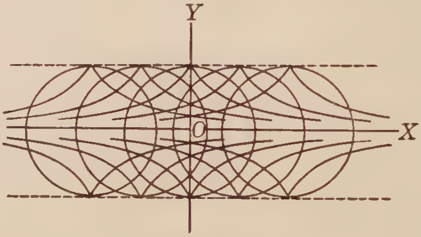

FIG. 93

As an example, consider the family of circles

$$(x - c)^2 + y^2 = a^2, \tag{4}$$

where c is an arbitrary constant and a is fixed (Fig. 93). The differential equation of the family is

$$p^2y^2 + y^2 = a^2. \tag{5}$$

Therefore the differential equation of the orthogonal family is

$$\frac{1}{p^2}y^2 + y^2 = a^2; \tag{6}$$

whence

$$\pm\frac{\sqrt{a^2 - y^2}}{y}\,dy = dx, \tag{7}$$

from which we have the family of tractrices

$$x - c = \pm\sqrt{a^2 - y^2} + a \log\frac{a \pm \sqrt{a^2 - y^2}}{y}. \tag{8}$$

It is evident that through any point of the plane between the lines $y = \pm a$ there go two circles and two tractrices, since both (5) and (6) are quadratic in p. These curves must be properly paired to show the orthogonal relation. For instance, if we take

$$p = + \frac{\sqrt{a^2 - y^2}}{y}$$

from (5), we must take

$$p = - \frac{y}{\sqrt{a^2 - y^2}}$$

from (6). Plotting these together we have the configuration shown in Fig. 94.

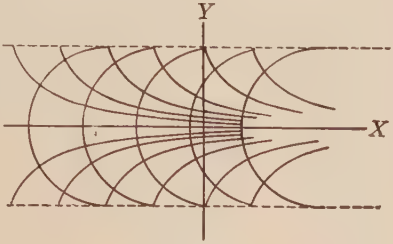

FIG. 94

98. Differential equation of the first order in three variables. Any equation of the form

$$f(x, y, z, c) = 0, \tag{1}$$

where c is an arbitrary constant, satisfies a differential equation of the form

$$P\,dx + Q\,dy + R\,dz = 0, \tag{2}$$

where P, Q, and R are functions of (x, y, z) but do not involve c. For from (1) we have

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0, \tag{3}$$

and the elimination of c from (3) and (1) gives (2).

This elimination may theoretically be carried out by solving equation (1) for c (§ 39), obtaining

$$\phi(x, y, z) = c; \tag{4}$$

whence

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0, \tag{5}$$

which must be the same equation as (2). Therefore either

$$\frac{\partial \phi}{\partial x} = P, \quad \frac{\partial \phi}{\partial y} = Q, \quad \frac{\partial \phi}{\partial z} = R \tag{6}$$

or

$$\frac{\partial \phi}{\partial x} = \mu P, \quad \frac{\partial \phi}{\partial y} = \mu Q, \quad \frac{\partial \phi}{\partial z} = \mu R. \tag{7}$$

In the first case equation (2) is exact; in the second case it has an integrating factor μ which makes it exact.

We now ask, conversely, if an equation (2) always has a solution of the form (1) or, what is the same thing, of the form (4). It is obvious from the foregoing that when this happens, P, Q, and R must satisfy either conditions (6) or conditions (7). Hence we make three cases for equation (2).

CASE I. *Exact equations.* Equation (2) is exact if a function ϕ exists for which conditions (6) hold. The necessary and sufficient conditions for this are (§ 36)

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}. \tag{8}$$

When these conditions are met, the function ϕ may be found by § 36, and the solution is then

$$\phi = c.$$

The solution may also be found by the method to be outlined in the next case.

The simplest case of an exact equation is that in which the variables are separated and the equation takes the form

$$f_1(x)dx + f_2(y)dy + f_3(z)dz = 0.$$

Conditions (8) are obviously met, and the solution is

$$\int f_1(x)dx + \int f_2(y)dy + \int f_3(z)dz = c.$$

CASE II. *Equations having integrating factors.* If equation (2) has an integrating factor μ, conditions (7) must be satisfied, and

$$\mu P\,dx + \mu Q\,dy + \mu R\,dz = 0$$

is exact. Therefore we must have

$$\frac{\partial(\mu P)}{\partial y}=\frac{\partial(\mu Q)}{\partial x}, \quad \frac{\partial(\mu Q)}{\partial z}=\frac{\partial(\mu R)}{\partial y}, \quad \frac{\partial(\mu R)}{\partial x}=\frac{\partial(\mu P)}{\partial z}. \tag{9}$$

Equations (9) may be written

$$\mu\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial \mu}{\partial x}-P\frac{\partial \mu}{\partial y},$$

$$\mu\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial \mu}{\partial y}-Q\frac{\partial \mu}{\partial z},$$

$$\mu\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial \mu}{\partial z}-R\frac{\partial \mu}{\partial x}.$$

Multiplying the first of these equations by R, the second by P, and the third by Q, and adding the resulting equations, we have (since, of course, $\mu \neq 0$)

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0, \tag{10}$$

which may be written in the symbolic form, easy to remember,

$$\begin{vmatrix} P & Q & R \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0. \tag{11}$$

This is, then, a necessary condition that must be satisfied in order that (2) should have an integrating factor. We shall prove that the condition is also sufficient by outlining a method of solution which will work when (10) is satisfied.

Suppose, then, that the coefficients of (2) satisfy condition (10). We will begin the solution of (2) by temporarily holding one of the variables constant. Let us choose to hold z constant. We have then

$$P\,dx+Q\,dy=0,$$

which will have a solution of the form

$$f(x, y, z)=c.$$

But c here means merely a constant as regards x and y. It therefore may be a function of z, and we write

$$f(x, y, z)=\phi(z). \tag{12}$$

We wish to determine ϕ so that (12) is a solution of (2). From equation (12) we have

$$f_x\,dx+f_y\,dy+(f_z-\phi')dz=0, \tag{13}$$

and if (12) solves (2), equation (13) must be the same as (2) except for a factor. Therefore

$$\begin{aligned} f_x &= \lambda P, \\ f_y &= \lambda Q, \\ f_z-\phi' &= \lambda R. \end{aligned} \tag{14}$$

The last equation in (14) may be written

$$f_z-\lambda R=\phi'. \tag{15}$$

If this equation contains on the left only z and ϕ, it is a differential equation to determine ϕ. Let us, then, solve (12) for y, thus,

$$y=F(x, z, \phi),$$

and make the substitution in $f_z - \lambda R$, the first member of (15). The necessary and sufficient condition that x should not appear in the result is that

$$\left[\frac{d(f_z - \lambda R)}{dx}\right]_{z,\,\phi} = 0, \tag{16}$$

where the expression on the left means the partial derivative when z and ϕ are constants. But by the laws of partial differentiation,

$$\left[\frac{d(f_z - \lambda R)}{dx}\right]_{z,\,\phi} = f_{zx} - \lambda\frac{\partial R}{\partial x} - R\frac{\partial \lambda}{\partial x} + \left(f_{zy} - \lambda\frac{\partial R}{\partial y} - R\frac{\partial \lambda}{\partial y}\right)F_x. \tag{17}$$

Now F_x means

$$\left(\frac{dy}{dx}\right)_{z,\,\phi},$$

and, from (12),

$$f_x\,dx + f_y\,dy + f_z\,dz = d\phi;$$

whence (§ 40)

$$F_x = -\frac{f_x}{f_y},$$

or, by use of (14),

$$F_x = -\frac{P}{Q}.$$

We assume that $Q \neq 0$, for it is obvious from (9) that if $Q = 0$ the equation (2) may be reduced to one which does not contain y. Hence we may place (17) in the form

$$Q\left[\frac{d(f_z - \lambda R)}{dx}\right]_{z,\,\phi} = Qf_{zx} - Pf_{zy} + \lambda\left(P\frac{\partial R}{\partial y} - Q\frac{\partial R}{\partial x}\right) + R\left(P\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial x}\right).$$

But, from (14),

$$f_{zx} = \lambda\frac{\partial P}{\partial z} + P\frac{\partial \lambda}{\partial z},$$

$$f_{zy} = \lambda\frac{\partial Q}{\partial z} + Q\frac{\partial \lambda}{\partial z},$$

$$\frac{\partial(\lambda P)}{\partial y} = \frac{\partial(\lambda Q)}{\partial x};$$

whence

$$P\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial x} = \lambda\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right),$$

and therefore

$$Q\left[\frac{d(f_z - \lambda R)}{dx}\right]_{z,\,\phi} = \lambda\left[P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right]. \tag{18}$$

Hence if condition (10) is satisfied, then equation (16) is satisfied, and equation (15) can be made an equation in ϕ, ϕ', and z

and can be solved for ϕ. This solution involves an arbitrary constant c. Then (12) is of the form

$$\Phi(x, y, z, c) = 0,$$

which satisfies (2). If (10) is not satisfied, this method of solution fails, as it should, since (10) is a necessary condition for the solution of (2).

Condition (10) is obviously satisfied if the equation is exact. Hence we may say:

The necessary and sufficient condition that the equation

$$P\,dx + Q\,dy + R\,dz = 0$$

may have a solution of the form

$$f(x, y, z, c) = 0$$

is that $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$

Geometrically we may say that the coefficients P, Q, R determine a vector

$$P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \tag{19}$$

at each point of space, and the differentials dx, dy, dz determine a vector

$$dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}. \tag{20}$$

The differential equation (2) asserts that these two vectors are perpendicular to each other. Hence the vector (20) is restricted to lie in a plane perpendicular to (19). In other words, the differential equation defines a plane of infinitesimal vectors (20) at each point of space. The totality of these vectors forms what we may call a planar element. The problem of integration is to arrange these planar elements into surfaces. This is possible only when the condition in the theorem is satisfied.

As an example of the practical application of the method of solution just outlined in theory, consider

$$yz^2\,dx + (y^2z - xz^2)dy - y^3\,dz = 0. \tag{21}$$

The condition for integrability is satisfied. To integrate, it is convenient to begin by holding y constant. We have then

$$yz^2\,dx - y^3\,dz = 0,$$

of which the solution is $x + \frac{y^2}{z} = \phi(y).$ (22)

Differentiating (22) we have

$$dx + \left(\frac{2y}{z} - \phi'\right)dy - \frac{y^2}{z^2}\,dz = 0. \tag{23}$$

Dividing equation (21) by yz^2 and comparing with (23) gives

$$\frac{y}{z} - \frac{x}{y} = \frac{2y}{z} - \phi'. \tag{24}$$

But, from (22), $$x = \phi - \frac{y^2}{z},$$

and therefore (24) is $$-\frac{\phi}{y} = -\phi',$$

or $$\frac{d\phi}{\phi} - \frac{dy}{y} = 0,$$

which gives $$\phi = cy.$$

Substituting this in (22) gives

$$x + \frac{y^2}{z} = cy,$$

or $$\frac{x}{y} + \frac{y}{z} = c,$$

as the solution of (21).

CASE III. *The nonintegrable case.* If condition (10) is not satisfied, the equation has no integral of the form

$$f(x, y, z, c) = 0,$$

and it is customary to say that the equation cannot be integrated. There is here a striking difference between the equation

$$M\,dx + N\,dy = 0$$

in two variables, which can always be integrated, and the similar equation in three or more variables.

Geometrically we may do something with the equation even in the nonintegrable case. As we have seen, equation (2) asserts that the direction $dx : dy : dz$ is perpendicular to the direction $P : Q : R$. To solve the equation is to determine geometric loci so that the condition of perpendicularity is fulfilled for directions on each locus. In the integrable case these loci consist of surfaces which are perpendicular at each point to the direction $P : Q : R$. Then any curve whatever drawn on the surface has this property of perpendicularity at each of its points.

In the nonintegrable case no such family of surfaces exists. We may, however, upon any surface whatever find a family of curves which has the property of being perpendicular to the direction $P:Q:R$. For if

$$F(x, y, z) = 0 \tag{25}$$

is any arbitrarily assumed surface, then any direction on this surface satisfies the equation

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz = 0. \tag{26}$$

This equation taken simultaneously with (2) defines a family of curves, as will be shown in the next section. These curves necessarily lie on (25).

For example, consider

$$xy\,dx + y\,dy + z\,dz = 0, \tag{27}$$

which is nonintegrable.

Assume the sphere $\qquad x^2 + y^2 + z^2 = a^2.$

Then

$$x\,dx + y\,dy + z\,dz = 0. \tag{28}$$

Taking (27) and (28) simultaneously, we have

$$dx = 0;$$

whence

$$x = c.$$

Hence the circles cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the planes $x = c$ satisfy (27) in a sense.

Again, still considering (27), assume the paraboloid

$$z = xy. \tag{29}$$

Then

$$y\,dx + x\,dy - dz = 0. \tag{30}$$

If (27) and (30) are taken simultaneously, we find that

$$xy\,dx + y\,dy + xy(y\,dx + x\,dy) = 0;$$

whence

$$(1 + y)\sqrt{1 + x^2} = c. \tag{31}$$

The curves defined by (29) and (31) satisfy (27).

99. Simultaneous equations in three variables. Let there be given two equations,

$$\begin{aligned} P_1\,dx + Q_1\,dy + R_1\,dz &= 0, \\ P_2\,dx + Q_2\,dy + R_2\,dz &= 0, \end{aligned} \tag{1}$$

where P_1, Q_1, R_1, P_2, Q_2, R_2 are functions of x, y, and z. These equations may be written in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \tag{2}$$

where $P = Q_1R_2 - Q_2R_1$, $Q = R_1P_2 - R_2P_1$, and $R = P_1Q_2 - P_2Q_1$. Accordingly we shall consider equations in the form (2).

Since $dx : dy : dz$ gives a direction in space, it seems graphically evident that the solution of (2) consists of a family of curves. Such a family is represented by two simultaneous equations of the form

$$\begin{aligned} f_1(x, y, z, c_1) &= 0, \\ f_2(x, y, z, c_2) &= 0. \end{aligned} \tag{3}$$

A proof will be given in the next section.

It is instructive to compare equations (2) with the equation

$$P\,dx + Q\,dy + R\,dz = 0, \tag{4}$$

discussed in § 98, considering P, Q, and R as the same in both (2) and (4). Equations (2) define a family of curves which everywhere have the direction of the vector

$$P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}. \tag{5}$$

If equation (4) has a solution, it defines a family of surfaces everywhere normal to the vector (5) and hence normal to the curves defined by (2). Now equations (2) always have a solution, but equation (4) does not.

Hence if a family of curves is given, it is not always possible to find a family of surfaces orthogonal to them. On the other hand, if a family of surfaces is given, P, Q, R are determined and equations (2) may be solved. Hence a family of curves may always be found orthogonal to a given family of surfaces.

Three methods of solution of equations (2) may be tried:

1. It may be possible to find two equations each of which contains only two variables and their differentials. For example, consider

$$\frac{dx}{xy} = \frac{dy}{y} = \frac{dz}{z}. \tag{6}$$

We readily find the two equations

$$\frac{dx}{x} = dy, \qquad \frac{dy}{y} = \frac{dz}{z},$$

the solutions of which are

$$y = c_1z, \qquad x = c_2e^y, \tag{7}$$

and equations (7) are the solutions of (6).

2. It may be possible to find readily one equation containing only two variables and their differentials. The solution of this equation may then be used to obtain another equation in two variables.

For example, consider

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xyze^x}. \tag{8}$$

From the equation

$$\frac{dx}{x} = \frac{dy}{y}$$

we find

$$y = c_1 x. \tag{9}$$

Taking the first and third fractions of (8) and using (9), we have

$$dx = \frac{dz}{c_1 xze^x};$$

whence

$$c_1(x - 1)e^x = \log c_2 x. \tag{10}$$

Then (10) and (9) taken simultaneously form the solution of (8). We may, if we like, eliminate c_1 from (10) and write the solution of (8) in the form

$$\begin{aligned} y &= c_1 x, \\ y(x-1)e^x &= x \log c_2 z. \end{aligned} \tag{11}$$

3. By the theory of fractions we may write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{k_1\,dx + k_2\,dy + k_3\,dz}{k_1P + k_2Q + k_3R}, \tag{12}$$

where k_1, k_2, k_3 are any multipliers, not necessarily constants, chosen at pleasure. In this way we may form new differential equations which may possibly be solved.

Particular interest attaches to the case in which k_1, k_2, k_3 can be so taken that

$$k_1P + k_2Q + k_3R = 0.$$

We then have the differential equation

$$k_1\,dx + k_2\,dy + k_3\,dz = 0,$$

which may perhaps be solved as in § 98.

As a first example, consider

$$\frac{dx}{x} = \frac{dy}{x+y} = \frac{dz}{x+z}. \tag{13}$$

We may write

$$\frac{dx}{x} = \frac{dy}{x+y} = \frac{dz}{x+z} = \frac{dy - dz}{y - z}.$$

The equation

$$\frac{dx}{x} = \frac{dy - dz}{y - z}$$

gives

$$x = c_1(y - z). \tag{14}$$

Using this in the equation

$$\frac{dy}{x+y}=\frac{dz}{x+z},$$

we get

$$\log(y-z)=c_2+\frac{y}{c_1(y-z)}. \tag{15}$$

Then (14) and (15) together form the solution of (13).

As a second example, consider

$$\frac{dx}{y+z}=\frac{dy}{-x}=\frac{dz}{x+y+z}. \tag{16}$$

We have, in the first place,

$$\frac{dx}{y+z}=\frac{dy}{-x}=\frac{dz}{x+y+z}=\frac{dx-dy-dz}{y+z-(-x)-(x+y+z)};$$

whence

$$dx-dy-dz=0,$$

and therefore

$$x-y-z=c_1. \tag{17}$$

Using (17) in the first fraction of (16), we have

$$\frac{dx}{x-c_1}=\frac{dy}{-x};$$

whence

$$x+c_1\log(x-c_1)=c_2-y, \tag{18}$$

and (17) and (18) taken together form the solution of (16).

100. Existence proof. Given

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}, \tag{1}$$

which are equivalent to

$$\begin{aligned}\frac{dy}{dx}&=\frac{Q}{P}=f_1(x,y,z),\\ \frac{dz}{dx}&=\frac{R}{P}=f_2(x,y,z).\end{aligned} \tag{2}$$

We assume that both $f_1(x, y, z)$ and $f_2(x, y, z)$ can be expanded into a power series in the neighborhood of (x_0, y_0, z_0), and shall take $x_0=0$, $y_0=0$, $z_0=0$.

Then

$$\begin{aligned}\frac{dy}{dx}&=\sum a_{ikl}x^iy^kz^l,\\ \frac{dz}{dx}&=\sum b_{ikl}x^iy^kz^l.\end{aligned} \tag{3}$$

Assume two power series,

$$y = a_1x + a_2x^2 + a_3x^3 + \cdots, \tag{4}$$

$$z = b_1x + b_2x^2 + b_3x^3 + \cdots, \tag{5}$$

and substitute in (3). It is easily seen that the unknown coefficients a_i and b_i are uniquely determined in terms of the known coefficients a_{ikl} and b_{ikl}, and that each successive coefficient a_i or b_i is expressed as a polynomial in the coefficients a_{ikl}, b_{ikl} and the coefficients a_i and b_i already obtained.

The series (4) and (5) are thus obtained. It remains to prove them convergent. For that purpose take a dominant function

$$\frac{M}{\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left(1-\frac{z}{c}\right)} \tag{6}$$

for each of the functions $f_1(x, y, z)$ and $f_2(x, y, z)$. In (6), (a, b, c) is a point at which each of the series in (3) converges absolutely, and M is a number which no coefficient in (3) can exceed. Consider then the differential equations

$$\begin{aligned} \frac{dy}{dx} &= \frac{M}{\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left(1-\frac{z}{c}\right)}, \\ \frac{dz}{dx} &= \frac{M}{\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left(1-\frac{z}{c}\right)}. \end{aligned} \tag{7}$$

If equations (7) are solved in power series

$$y = A_1x + A_2x^2 + \cdots, \tag{8}$$

$$z = B_2x + B_2x^2 + \cdots, \tag{9}$$

in the same manner as equations (3) were solved, the manner in which these coefficients are found shows that

$$|A_i| > |a_i|, \qquad |B_i| > |b_i|,$$

and hence if (8) and (9) converge, so will (4) and (5).

Now (7) may be solved in an elementary manner. It is evident that $y = z$, and therefore $c = b$, and we have only to solve

$$\left(1-\frac{y}{b}\right)^2 dy = \frac{M\,dx}{1-\frac{x}{a}};$$

whence

$$y = b - b\sqrt[3]{\frac{3\,aM}{b}\log\left(1-\frac{x}{a}\right)}. \tag{10}$$

This may be expanded into a convergent series which can be none other than (8). The series (9) is the same, since $z = y$. Hence, as shown above, the convergence of (4) and (5) is proved.

For convenience we have used (0, 0, 0) for (x_0, y_0, z_0). Replacing (x_0, y_0, z_0), we may write (4) and (5) as

$$\begin{aligned} y &= y_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots, \\ z &= z_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots. \end{aligned} \tag{11}$$

Here the coefficients y_0, z_0 are purely arbitrary, and the other coefficients are determined by them and the coefficients of the given equation.

We see, then, that equations (1) have solutions involving two arbitrary constants. The most general form is to write these as

$$\begin{aligned} f(x, y, z, c_1, c_2) &= 0, \\ F(x, y, z, c_1, c_2) &= 0. \end{aligned} \tag{12}$$

These solved for y and z are equivalent to (8) and (9), and the constants are determined if we know that for $x = x_0$, y becomes y_0 and z becomes z_0.

The methods we have used are evidently extensible to the equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n},$$

and we say that these equations have solutions of the form

$$f_i(x_1, x_2, \cdots, x_n, c_1, c_2, \cdots c_n) = 0,$$

where there are $(n - 1)$ functions f_i involving $n - 1$ arbitrary constants.

EXERCISES

Solve by one of the first seven methods of § 91:

1. $\tan x \tan y \, dx + \sec^2 y \, dy = 0.$
2. $(1 + x^2)y \, dx + (1 - y^2)x \, dy = 0.$
3. $xy(1 + x^2)dy - (1 - y^2)dx = 0.$
4. $(x + y)dx + x \, dy = 0.$
5. $(y - \sqrt{x^2 + y^2})dx - x \, dy = 0.$ homog
6. $\left(x \sin \frac{y}{x} + y \cos \frac{y}{x}\right)dx - x \cos \frac{y}{x} \, dy = 0.$ ho
7. $(x + 2y - 3)dx + (2x - y - 1)dy = 0.$ x
8. $(x + y)dx + (x + y + 1)dy = 0.$ exact

9. $dy + (y - \cos x)dx = 0.$

10. $x\,dy + (y - xe^x)dx = 0.$

11. $(1 + x^2)\dfrac{dy}{dx} - xy = x(1 + x^2).$

12. $2\,y\dfrac{dy}{dx} - 2\,y^2 = x^2 + 1.$

13. $(1 + x^2)\dfrac{dy}{dx} - y + y^2 = 0.$

14. $x^2(1 + x^2)\dfrac{dy}{dx} - x^3y = y^3.$

15. $\dfrac{1 - y^2}{x^3}\,dx + \dfrac{1 + x^2}{x^2}\,y\,dy = 0.$

16. $(x - y)^2\,dx - (x^2 - 2\,xy + 3\,y^2)dy = 0.$

17. $(x + y + x^2y)dx + (x + x^3)dy = 0.$

18. $ye^{-\frac{x}{y}}\,dx - (xe^{-\frac{x}{y}} + y^2)dy = 0.$

19. $(xy^2 + y^2)dx + (x^2y - x)dy = 0.$

20. $(x^3 + y^3)dx - xy^2\,dy = 0.$

Solve the following equations by series:

21. $\dfrac{dy}{dx} = ky.$

22. $\dfrac{dy}{dx} = kx.$

23. $\dfrac{dy}{dx} = xy.$

24. $\dfrac{dy}{dx} = x + y^2.$

25. $\dfrac{dy}{dx} = x^3 + y^2.$

26. $\dfrac{dy}{dx} = y^2 - x.$

Solve the following equations by the methods of § 92:

27. $xy(p^2 + 1) - (x^2 + y^2)p = 0.$

28. $x^2p^2 + xyp - 2\,y^2 = 0.$

29. $y(1 - p^2) - 2\,px = 0.$

30. $y = 2\,px - p^2y.$

31. $p^3 + 4\,xyp - 8\,y^2 = 0.$

32. $x^2p^2 - p^2 + 1 = 0.$

33. $py^2 - 2\,p^2xy + p^3x^2 = 1.$

34. $(1 + y^2)p^2 - 2\,xyp^3 + x^2p^4 = 1.$

In Exercises 35–38 show that the differential equation of the curve in each case is a Clairaut equation, and find the curve:

35. A curve such that each tangent makes intercepts on the coördinate axis whose sum is k.

36. A curve in which the projection upon OY of the perpendicular from the origin to any tangent is k.

37. A curve in which the portion of the tangent between the coördinate axes has the length k.

38. A curve such that the area between the tangent and the coördinate axes is k^2.

39. Find the envelope of the family of straight lines $y = 2\,cx + c^4$.

40. Find the envelope of the family of parabolas $y^2 = c(x - c)$.

41. The semi-axes of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are such that $ab = k^2$, where k^2 is constant. Find the envelope of the family.

42. Find the envelope of the straight lines $y = cx - 2ac - ac^3$.

43. Find the envelope of the family of circles having their centers on the line $y = 2x$ and tangent to the axis of y.

44. Find the envelope of the family of circles which have their centers on the parabola $y^2 = 4ax$ and pass through the vertex of the parabola.

45. If rays of light emanating from a fixed point in a plane are reflected from a curve, the envelope of the reflected rays is a *caustic curve.* Show that the caustic curve of rays issuing from a point on a circle and reflected by the circle is a cardioid.

46. Find the evolute of the parabola $y^2 = 4ax$.

47. Show that the evolute of a tractrix is a catenary.

48. Show that the evolute of a cycloid is an equal cycloid.

49. Find the evolute of an ellipse

(1) from the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

(2) from the equations $x = a\cos\phi$, $y = b\sin\phi$.

50. Find the evolute of the four-cusped hypocycloid $x = a\cos^3\phi$, $y = a\sin^3\phi$.

51. Find the orthogonal trajectories of the family of parabolas $y^2 = 4cx$.

52. Find the orthogonal trajectories of the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 + c} = 1,$$

c being the variable parameter.

53. Find the orthogonal trajectories of the confocal parabolas $y^2 = 4cx + 4c^2$.

54. Find the orthogonal trajectories of the family of ellipses in which the minor axis is one half the major axis.

55. Find the orthogonal trajectories of the family of circles each of which passes through the same fixed points.

56. Find the orthogonal trajectories of the family of circles each of which is tangent to the same straight line at the same point.

57. If $f\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ is the equation of a family of curves in polar coördinates, prove that $f\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0$ is the equation of their orthogonal trajectories.

58. Find the orthogonal trajectories of the family of lemniscates $r^2 = 2c^2\cos 2\theta$.

59. Find the orthogonal trajectories of the family of cardioids $r = c(\cos\theta + 1)$.

60. Find the orthogonal trajectories of the family of logarithmic spirals $r = e^{c\theta}$.

61. Find the singular solution of $(x^2 - a^2)p^2 - 2\,xyp - x^2 = 0$.

62. Find the singular solution of $y = -xp + x^4p^2$.

63. Find the singular solution of $a^2 - y^2 = p^2y^2$.

64. Find the singular solution of $x^3p^2 + x^2yp + a^3 = 0$.

65. Find the singular solution of $p^3 - 4\,xyp + 8\,y^2 = 0$.

Solve the following equations:

66. $(y + z - b - c)dx + (z + x - c - a)dy + (x + y - a - b)dz = 0$.

67. $\left(\frac{1}{y} - \frac{z}{x^2}\right)dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy + \left(\frac{1}{x} - \frac{y}{z^2}\right)dz = 0$.

68. $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0$.

69. $yz^2\,dx + (y^2z - xz^2)dy - y^2(y + z)dz = 0$.

70. $yz\,dx - zx\,dy + (x^2 + y^2)dz = 0$.

Solve the following systems of simultaneous equations:

71. $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$.

72. $\frac{dx}{y} = \frac{dy}{x + z} = \frac{dz}{y}$.

73. $\frac{dx}{z} = \frac{x\,dy}{x^2 + z^2} = -\frac{dz}{z}$.

74. $\frac{dx}{x + y - z} = \frac{dy}{z} = dz$.

75. $\frac{dx}{x^2 + y^2} = \frac{dy}{2\,xy} = \frac{dz}{(x - y)z}$.

76. $\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y}$.

77. $\frac{dx}{x - y - z} = \frac{dy}{y - z - x} = \frac{dz}{z}$.

78. $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2\,xy} = \frac{dz}{2\,xz}$.

CHAPTER XI

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

101. Existence proof. Consider the equation

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \cdots, \frac{d^{n-1} y}{dx^{n-1}}\right). \tag{1}$$

This may be replaced by the group of equations

$$\begin{aligned} \frac{dy}{dx} &= y_1, \\ \frac{dy_1}{dx} &= y_2, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{dy_{n-2}}{dx} &= y_{n-1}, \\ \frac{dy_{n-1}}{dx} &= f(x, y, y_1, y_2, \cdots, y_{n-1}). \end{aligned} \tag{2}$$

By the previous section this group has a solution which for $x = x_0$ reduces to $y = y_0$, $y_1 = (y_1)_0$, $y_2 = (y_2)_0$, $\cdots$, $y_{n-1} = (y_{n-1})_0$, values which may be arbitrarily assigned. Hence in (1) the values of $y, \frac{dy}{dx}, \cdots, \frac{dy_{n-1}}{dx_{n-1}}$ may be arbitrarily assigned and the equation solved. This is expressed by saying that the general solution of (1) involves n arbitrary constants.

The existence of the solution having thus been shown, let us substitute in (1) the series

$$y = a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

where x_0 and y_0 are assumed as zero.

Then

$$\begin{aligned} y_1 &= a_1 + 2\,a_2 x + 3\,a_3 x^2 + \cdots, \\ y_2 &= 2\,a_2 + 6\,a_3 x + 12\,a_4 x^2 + \cdots, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_{n-1} &= (n-1)!\,a_{n-1} + n!\,a_n x + \cdots. \end{aligned}$$

All equations in (2) but the last are now satisfied, and the coefficients a_i are obtained by substituting in the last equation in (2). The series expansion may thus be practically obtained.

Another method is as follows: Equation (1) gives us $\frac{d^ny}{dx^n}$ in terms of x, y, $\frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}$. By differentiating and substituting we may find $\frac{d^{n+1}y}{dx^{n+1}}, \frac{d^{n+2}y}{dx^{n+2}}, \cdots$ in terms of the same variables. Assuming, then, x_0, y_0, $\left(\frac{dy}{dx}\right)_0, \cdots, \left(\frac{dy^{n-1}}{dx^{n-1}}\right)_0$ at pleasure, we compute $\left(\frac{d^ny}{dx^n}\right)_0, \left(\frac{d^{n+1}y}{dx^{n+1}}\right)_0, \cdots$, and may then write down Taylor's expansion for y as a power series in $x - x_0$.

102. The linear differential equation. The equation

$$\frac{d^ny}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_{n-1} \frac{dy}{dx} + p_n y = R, \quad (1)$$

where $p_1, p_2, \cdots, p_n$, and R are functions of x, or constants, is a linear differential equation of the nth order. Equation (1) in which $R \neq 0$ is called the *complete equation* in distinction to the equation

$$\frac{d^ny}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1} \frac{dy}{dx} + p_n y = 0, \quad (2)$$

which is called the *reduced equation.* The solution of (1) is closely connected with the solution of (2). This will be brought out in the theorems which we shall now prove:

I. If $y_1, y_2, \cdots, y_n$ are n linearly independent solutions of the reduced equation (2), *the general solution of the reduced equation is*

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n. \quad (3)$$

That (3) solves (2) is evident from direct substitution; that it is the general solution follows from the fact that it contains n arbitrary constants under the hypothesis that the functions y_i are linearly independent.

The necessity of this hypothesis is seen from an example. Consider a cubic equation

$$\frac{d^3y}{dx^3} + p \frac{d^2y}{dx^2} + q \frac{dy}{dx} + ry = 0, \quad (4)$$

and let y_1, y_2, y_3 be three solutions. Then

$$y = c_1y_1 + c_2y_2 + c_3y_3 \quad (5)$$

is also a solution and will be the general solution if y_1, y_2, y_3 are linearly independent. But if, for example,

$$y_3 = ay_1 + by_2,$$

then (5) becomes $\quad y = (c_1 + a)y_1 + (c_2 + b)y_2,$

which involves only two arbitrary constants, namely, $C_1 = c_1 + a$ and $C_2 = c_2 + a$, and is therefore not the general solution.

II. If I_1 *and* I_2 *are any two particular solutions of* (1), *they differ by terms of the form*

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

where $y_1, y_2, \cdots, y_n$ *are solutions of* (2).

To prove this, note that by hypothesis

$$\frac{d^nI_1}{dx^n} + p_1\frac{d^{n-1}I_1}{dx^{n-1}} + \cdots + p_{n-1}\frac{dI_1}{dx} + p_nI_1 = R,$$

$$\frac{d^nI_2}{dx^n} + p_1\frac{d^{n-1}I_2}{dx^{n-1}} + \cdots + p_{n-1}\frac{dI_2}{dx} + p_nI_2 = R;$$

whence by subtraction $I_1 - I_2$ satisfies equation (2), and therefore, by theorem I,

$$I_1 - I_2 = c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

as was to be proved.

III. The general solution of (1) *is of the form*

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + I, \tag{6}$$

where $c_1y_1 + c_2y_2 + \cdots + c_ny_n$ *is the general solution of* (2), *and* I *is any particular integral of* (1).

This is an immediate consequence of II. It may also be shown by direct substitution in (1) that (6) satisfies (1). That it is the general solution follows from the fact that it contains n arbitrary constants.

In the solution (6) the quantity I is called the *particular integral*, and the part involving the arbitrary constants is called the *complementary function.*

These theorems give certain general facts of importance regarding the solutions of (1) and (2), and use will be made of them in the subsequent sections. We may at times find use for the following theorem:

IV. If one solution of the reduced equation is known, the degree of that equation may be lowered by unity.

Let y_1 be a solution of the reduced equation (2) and substitute

$$y = y_1 z.$$

It is not difficult to see that the coefficient of z in the new equation is the same as the left-hand member of (2), with y replaced by y_1, and therefore vanishes, since y_1 is a solution of (2). The new equation is then of the form

$$\frac{d^n z}{dx^n} + q_1 \frac{d^{n-1} z}{dx^{n-1}} + \cdots + q_{n-2} \frac{d^2 z}{dx^2} + q_{n-1} \frac{dz}{dx} = 0. \tag{7}$$

If we now place $\frac{dz}{dx} = u$, we have a linear differential equation of the $(n-1)$st order,

$$\frac{d^{n-1} u}{dx^{n-1}} + q_1 \frac{d^{n-2} u}{dx^{n-2}} + \cdots + q_{n-2} \frac{du}{dx} + q_{n-1} u = 0, \tag{8}$$

which proves the proposition.

If y_2 is another known solution of (2), then $z = \frac{y_2}{y_1}$ is a solution of (7), and $\frac{d}{dx}\left(\frac{y_2}{y_1}\right)$ is a solution of (8). Hence the degree of equation (8) may be reduced by unity. Proceeding in this way we have the following theorem:

V. If m solutions of the reduced equation are known, the solution of that equation may be reduced to the solution of an equation of degree $n - m$.

103. Method of variation of constants. We shall in this section present a method by which if the solution of the reduced equation is known that of the complete equation may be found. For simplicity of treatment we will take an equation of the third order,

$$\frac{d^3 y}{dx^3} + p \frac{d^2 y}{dx^2} + q \frac{dy}{dx} + ry = R, \tag{1}$$

and suppose that we have found the solution of

$$\frac{d^3 y}{dx^3} + p \frac{d^2 y}{dx^2} + q \frac{dy}{dx} + ry = 0 \tag{2}$$

in the form

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3. \tag{3}$$

In (3) c_1, c_2, and c_3 are constants and (2) is satisfied. The question now is, May we not replace c_1, c_2, c_3 by functions of x in such a way that (1) is satisfied? We will therefore consider the c_i as variables. For this reason the method is called that of the variation of constants, though it might more properly be called the replacement of constants by variables.

If we take c_i as functions of x, we find from (3) that

$$\frac{dy}{dx} = c_1 \frac{dy_1}{dx} + c_2 \frac{dy_2}{dx} + c_3 \frac{dy_3}{dx} + c_1'y_1 + c_2'y_2 + c_3'y_3, \tag{4}$$

where $$c_1' = \frac{dc_1}{dx}, \quad c_2' = \frac{dc_2}{dx}, \quad c_3' = \frac{dc_3}{dx}.$$

Since there are three functions c_i to be determined, we may impose three conditions upon them. We take the first to be

$$c_1'y_1 + c_2'y_2 + c_3'y_3 = 0. \tag{5}$$

Then (4) becomes

$$\frac{dy}{dx} = c_1 \frac{dy_1}{dx} + c_2 \frac{dy_2}{dx} + c_3 \frac{dy_3}{dx}. \tag{6}$$

Differentiating again,

$$\frac{d^2y}{dx^2} = c_1 \frac{d^2y_1}{dx^2} + c_2 \frac{d^2y_2}{dx^2} + c_3 \frac{d^2y_3}{dx^2} + c_1' \frac{dy_1}{dx} + c_2' \frac{dy_2}{dx} + c_3' \frac{dy_3}{dx}. \tag{7}$$

We take $$c_1' \frac{dy_1}{dx} + c_2' \frac{dy_2}{dx} + c_3' \frac{dy_3}{dx} = 0 \tag{8}$$

as the second condition to be imposed upon the c's. We thus reduce equation (7) to the form

$$\frac{d^2y}{dx^2} = c_1 \frac{d^2y_1}{dx^2} + c_2 \frac{d^2y_2}{dx^2} + c_3 \frac{d^2y_3}{dx^2}; \tag{9}$$

whence

$$\frac{d^3y}{dx^3} = c_1 \frac{d^3y_1}{dx^3} + c_2 \frac{d^3y_2}{dx^3} + c_3 \frac{d^3y_3}{dx^3} + c_1' \frac{d^2y_1}{dx^2} + c_2' \frac{d^2y_2}{dx^2} + c_3' \frac{d^2y_3}{dx^2}. \tag{10}$$

Substituting (3), (6), (9), and (10) in (1), we have

$$c_1' \frac{d^2y_1}{dx^2} + c_2' \frac{d^2y_2}{dx^2} + c_3' \frac{d^2y_3}{dx^2} = R. \tag{11}$$

Equations (5), (8), (11) are now three linear equations which may be solved by elementary algebra for c_1', c_2', c_3'. Then, by integration, $$c_1 = \phi_1(x), \quad c_2 = \phi_2(x), \quad c_3 = \phi_3(x).$$

Hence $$I = \phi_1(x)y_1 + \phi_2(x)y_2 + \phi_3(x)y_3$$

is a solution of (1). Therefore the general solution is, by III, § 102,

$$y = [c_1 + \phi_1(x)]y_1 + [c_2 + \phi_2(x)]y_2 + [c_3 + \phi_3(x)]y_3. \tag{12}$$

104. The linear differential equation with constant coefficients. In this section we shall consider the equation

$$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x), \tag{1}$$

where the coefficients a_i are constants. It is convenient to express $\frac{dy}{dx}$ by Dy, $\frac{d^2y}{dx^2}$ by $D^2y, \cdots,$ $\frac{d^ny}{dx^n}$ by D^ny, and to write (1) in the form

$$D^ny + a_1D^{n-1}y + \cdots + a_{n-1}Dy + a_ny = f(x),$$

or, more compactly,

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = f(x). \tag{2}$$

The expression in parenthesis preceding y in (2) is called an *operator*, and we are said to operate on a quantity with it when we carry out the indicated operations of differentiation, multiplication, and addition. The solution of (2) is expressed by the symbol

$$y = \frac{1}{D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n} f(x), \tag{3}$$

where the expression on the right is not to be considered as a fraction, but merely as the inverse operator to that denoted by the operator in (2).

Let us now treat the operator as if it were an algebraic polynomial in which D is a quantity instead of a symbol of differentiation, and split it up into linear factors, writing the left-hand member of (2) as

$$(D - r_1)(D - r_2) \cdots (D - r_{n-1})(D - r_n)y. \tag{4}$$

If we consider that (4) means to operate first on y with $D - r_n$, on the result obtained with $D - r_{n-1}$, on that result with $D - r_{n-2}$, and so on, we assert that (4) is exactly equivalent to the operation on the left-hand side of (2). This follows from the fact that D considered as an operator obeys the same laws as when it is considered as an algebraic quantity. The student may verify this by considering

$$(D - r_1)(D - r_2)y.$$

This is first of all equivalent to

$$(D - r_1)(Dy - r_2y),$$

and this to

$$D(Dy - r_2y) - r_1(Dy - r_2y) = [D^2 - (r_1 + r_2)D + r_1r_2]y.$$

Similarly,

$$\begin{aligned}&(D-r_1)(D-r_2)(D-r_3)y\\&\quad=(D-r_1)[D^2-(r_2+r_3)D+r_2r_3]y\\&\quad=[D^3-(r_1+r_2+r_3)D^2+(r_1r_2+r_2r_3+r_3r_1)D-r_1r_2r_3]y,\end{aligned}$$

and so on.

A study of these results and the similar ones to be obtained with more factors shows that the order of the factors is immaterial.

We have, then, reduced equation (1) to the form

$$(D-r_1)(D-r_2)\cdots(D-r_{n-1})(D-r_n)y=f(x). \tag{5}$$

The simplest equation of the form (1) or (5) occurs when $n=1$, and we have

$$(D-r_1)y=f(x), \tag{6}$$

the solution of which is, by Case IV, § 91,

$$y=e^{r_1x}\int e^{-r_1x}f(x)dx+c_1e^{r_1x}. \tag{7}$$

We have, accordingly, the formula

$$\frac{1}{D-r_1}f(x)=e^{r_1x}\int e^{-r_1x}f(x)dx, \tag{8}$$

where the constant of integration may be supplied or not in evaluating the integral. If the constant is used, (8) gives the general solution; if the constant is not used, we have a particular integral.

Returning to equation (5), it is clear that we have in succession

$$\begin{aligned}(D-r_2)(D-r_3)\cdots(D-r_n)y&=\frac{1}{D-r_1}f(x)\\&=e^{r_1x}\int e^{-r_1x}f(x)dx,\\(D-r_3)\cdots(D-r_n)y&=\frac{1}{D-r_2}\left[e^{r_1x}\int e^{-r_1x}f(x)dx\right]\\&=e^{r_2x}\int e^{(r_1-r_2)x}\int e^{-r_1x}f(x)dx^2,\end{aligned}$$

and so on. Hence, finally,

$$y=e^{r_nx}\int e^{(r_{n-1}-r_n)x}\int\cdots\int e^{(r_1-r_2)x}\int e^{-r_1x}f(x)dx^n. \tag{9}$$

Equation (9) furnishes a general formula for the solution of equation (1) or (5). As each integration is performed, a constant of integration may be introduced so that the solution contains n

arbitrary constants and is therefore the general solution, or the constants may be omitted and supplied at the end by theorem III, § 102.

The formula, however, is frequently tedious in its application, and we shall present in the next sections more convenient methods of solution for the cases most often arising in practice.

105. The complementary function. All the theorems of § 102 hold for the linear differential equation with constant coefficients. We shall therefore concern ourselves first with the reduced equation

$$(D - r_1)(D - r_2) \cdots (D - r_n)y = 0. \tag{1}$$

Since the left-hand member of (1) is independent of the order of the factors, we may write any factor $D - r_k$ in the place next to the y. Then if

$$(D - r_k)y = 0, \tag{2}$$

equation (1) is satisfied, since all the operations on zero give zero. A solution of (2) is accordingly

$$y = c_k e^{r_k x}, \tag{3}$$

where c_k is an arbitrary constant. Giving k all values from 1 to n, we have

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}. \tag{4}$$

If all the r's are distinct, this will be the general solution of (1) by theorem I, § 102. If, however, some of the r's are equal, the expression (4), although a solution of (1), will not be the general solution, since it will not contain n arbitrary constants. Suppose, for example, that the first k values of r are equal. Then the first k terms of (4) are $(c_1 + c_2 + \cdots + c_k)e^{r_1 x}$, which is equivalent simply to $Ce^{r_1 x}$.

In this case equation (1) may be written

$$(D - r_{k+1}) \cdots (D - r_n)(D - r_1)^k y = 0,$$

and any solution of

$$(D - r_1)^k y = 0 \tag{5}$$

is a solution of (1). The solution of (5) may be found by the general formula (9) of § 104. We have

$$\begin{aligned} y &= e^{r_1 x} \int\!\!\int \cdots \int 0 \, dx^k \\ &= (c_1 + c_2 x + \cdots + c_k x^{k-1})e^{r_1 x}, \end{aligned} \tag{6}$$

which now replaces the first k terms of (4).

Formula (4) with its modification (6) is perfectly general irrespective of the nature of the quantities r_i. If the coefficients of (1) are real, then, by a well-known theorem of algebra, all imaginary values of r_i will appear in pairs as conjugate imaginary. The corresponding terms in (4) or (6) may then be modified so as to appear in a real form. Suppose, for clearness, that

$$r_1 = \lambda + \mu i, \qquad r_2 = \lambda - \mu i,$$

where λ and μ are real quantities and $i = \sqrt{-1}$.

We then have

$$\begin{aligned} c_1e^{r_1x} + c_2e^{r_2x} &= e^{\lambda x}(c_1e^{\mu ix} + c_2{}^{-\mu ix}) \\ &= e^{\lambda x}[(c_1 + c_2)\cos \mu x + i(c_1 - c_2)\sin \mu x], \end{aligned}$$

the last transformation being made by § 26.

Let us now place

$$c_1 + c_2 = C_1, \qquad i(c_1 - c_2) = C_2,$$

and we have

$$c_1e^{r_1x} + c_2e^{r_2x} = e^{\lambda x}(C_1 \cos \mu x + C_2 \sin \mu x). \tag{7}$$

Similarly, if the factor $D - (\lambda + \mu i)$ occurs k times, so does the factor $D - (\lambda - \mu i)$, and we have, from (6),

$$e^{\lambda x}(c_1 + c_2x + \cdots + c_kx^{k-1})e^{\mu ix} + e^{\lambda x}(c_1' + c_2'x + \cdots + c_k'x^{k-1})e^{-\mu ix},$$

which is equivalent to

$$\begin{aligned} e^{\lambda x}[(C_1 + C_2x + \cdots + C_kx^{k-1})\cos \mu x + (B_1 + B_2x + \cdots \\ + B_kx^{k-1})\sin \mu x]. \end{aligned} \tag{8}$$

106. The particular integral. Consider now the complete equation

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = f(x). \tag{1}$$

The complementary function is to be found as in § 105. The particular integral may be found by the method of variation of constants or by applying formula (9), § 104, omitting the constants of integration. Such methods are frequently tedious. A more convenient way is to assume the form which the particular integral will take, using undetermined coefficients, which are then determined by substituting in the equation. The form of the solution may be inferred by studying the results obtained by (9), § 104, for different functions $f(x)$. There result in certain common cases the following rules, in which we denote the differential equation by

$$P(D)y = f(x),$$

where $P(D)$ is a polynomial in D, and denote the particular integral by I.

I. If $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, *assume, in general,*

$$I = Ax^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n;$$

but if D^m *is a factor of* $P(D)$, *assume*

$$I = x^m(Ax^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n).$$

II. If $f(x) = ce^{ax}$, *assume, in general,*

$$I = Ae^{ax};$$

but if $(D-a)^m$ *is a factor of* $P(D)$, *assume*

$$I = Ax^me^{ax}.$$

III. If $f(x) = c \sin ax$ *or* $c \cos ax$, *assume, in general,*

$$I = A \sin ax + B \cos ax;$$

but if $(D^2 + a^2)^m$ *is a factor of* $P(D)$, *assume*

$$I = x^m(A \sin ax + B \cos ax).$$

IV. If $f(x) = e^{ax}\phi(x)$, *place* $y = e^{ax}z$ *and divide out* e^{ax}.

V. If $f(x)$ *is the sum of a number of functions, take I as the sum of the particular integrals corresponding to each of the functions.*

Example 1. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = e^{4x}.$

This may be written $(D+3)(D-2)y = e^{4x}$.

The complementary function is $c_1e^{2x} + c_2e^{-3x}$. To find the particular integral we place

$$I = Ae^{4x}$$

and substitute in the equation. We obtain

$$14\,Ae^{4x} = e^{4x};$$

whence

$$A = \tfrac{1}{14}.$$

Therefore the general solution is

$$y = c_1e^{2x} + c_2e^{-3x} + \tfrac{1}{14}e^{4x}.$$

Example 2. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = \sin 2x.$

This may be written $D^2(D+1)y = \sin 2x$.

The complementary function is therefore $c_1 + c_2x + c_3e^{-x}$. To find the particular integral we place

$$I = A \sin 2x + B \cos 2x$$

and substitute in the equation. We obtain

$$(8B - 4A) \sin 2x - (4B + 8A) \cos 2x = \sin 2x.$$

To satisfy the equation we must have

$$8B - 4A = 1, \qquad 4B + 8A = 0;$$

whence $$B = \tfrac{1}{10}, \qquad A = -\tfrac{1}{20}.$$

Therefore the general solution is

$$y = c_1 + c_2 x + c_3 e^{-x} - \tfrac{1}{20} \sin 2x + \tfrac{1}{10} \cos 2x.$$

Example 3. $\dfrac{d^2y}{dx^2} + y = \sin x.$

This may be written $(D^2 + 1)y = \sin x$.

By III, we write $I = Ax \sin x + Bx \cos x$ and substitute in the equation. There results

$$-2B \sin x + 2A \cos x = \sin x.$$

Therefore $$B = -\tfrac{1}{2}, \qquad A = 0.$$

The general solution is

$$y = c_1 e^{ix} + c_2 e^{-ix} - \frac{x}{2} \cos x = C_1 \cos x + C_2 \sin x - \frac{x}{2} \cos x.$$

107. Equations reducible to linear equations with constant coefficients. Consider the equation

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = f(x), \qquad (1)$$

where $a_1, a_2, \cdots, a_{n-1}, a_n$ are constants. This equation has the peculiarity that each derivative is multiplied by a power of x equal to the order of the derivative. It can be reduced to a linear equation with constant coefficients by placing

$$x = e^z.$$

Then $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^{-z}\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)\frac{dz}{dx} = e^{-2z}\frac{d^2y}{dz^2} - e^{-2z}\frac{dy}{dz},$$

$$\frac{d^3y}{dx^3} = \frac{d}{dz}\left(e^{-2z}\frac{d^2y}{dz^2} - e^{-2z}\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= e^{-3z}\frac{d^3y}{dz^3} - 3e^{-3z}\frac{d^2y}{dz^2} + 2e^{-3z}\frac{dy}{dz},$$

and so on.

Hence
$$x\frac{dy}{dx} = Dy,$$
$$x^2\frac{d^2y}{dx^2} = (D^2 - D)y,$$
$$x^3\frac{d^3y}{dx^3} = (D^3 - 3D^2 + 2D)y,$$
and so on, where $D = \frac{d}{dz}$.

For example, the equation
$$x^3\frac{d^3y}{dx^3} + 5x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} = x^2$$
becomes
$$(D^3 + 2D^2)y = e^{2z};$$
whence
$$y = c_1 + c_2z + c_3e^{-2z} + \tfrac{1}{16}e^{2z}$$
$$= c_1 + c_2\log x + \frac{c_3}{x^2} + \frac{1}{16}x^2.$$

108. Simultaneous linear differential equations with constant coefficients. The operators of the previous sections may be employed in solving a system of two or more linear differential equations with constant coefficients when the equations involve only one independent variable and a number of dependent variables equal to the number of the equations. The method by which this may be done can best be explained by an example.

Consider
$$\frac{dx}{dt} + \frac{dy}{dt} - x - 4y = e^{5t},$$
$$\frac{dx}{dt} + \frac{dy}{dt} - 2x - 3y = e^{2t}.$$

These equations may be written
$$(D-1)x + (D-4)y = e^{5t}, \qquad (1)$$
$$(D-2)x + (D-3)y = e^{2t}. \qquad (2)$$

We may now eliminate y from the equations in a manner analogous to that used in solving two algebraic equations. We first operate on (1) with $D-3$, the coefficient of y in (2), and have
$$(D^2 - 4D + 3)x + (D^2 - 7D + 12)y = 2e^{5t}, \qquad (3)$$

since $(D-3)e^{5t} = 5\,e^{5t} - 3\,e^{5t} = 2\,e^{5t}$. We then operate on equation (2) with $D-4$, the coefficient of y in (1), and have

$$(D^2 - 6\,D + 8)x + (D^2 - 7\,D + 12)y = -2\,e^{2t}, \tag{4}$$

since $(D-4)e^{2t} = -2\,e^{2t}$. By subtracting (4) from (3) we have

$$(2\,D - 5)x = 2\,e^{5t} + 2\,e^{2t}, \tag{5}$$

the solution of which is

$$x = c_1 e^{\frac{5}{2}t} + \tfrac{2}{5}\,e^{5t} - 2\,e^{2t}. \tag{6}$$

Similarly, by operating on (1) with $D-2$ and on (2) with $D-1$, and subtracting the result of the first operation from that of the second, we have

$$(2\,D - 5)y = -3\,e^{5t} + e^{2t}, \tag{7}$$

the solution of which is

$$y = c_2 e^{\frac{5}{2}t} - \tfrac{3}{5}e^{5t} - e^{2t}. \tag{8}$$

The constants in (6) and (8) are, however, not independent, for if the values of x and y given in (6) and (8) are substituted in the original equations (1) and (2), they must reduce the latter equations to identities. Making these substitutions, we have

$$\tfrac{3}{2}(c_1 - c_2)e^{\frac{5}{2}t} + e^{5t} = e^{5t},$$

and

$$\tfrac{1}{2}(c_1 - c_2)e^{\frac{5}{2}t} + e^{2t} = e^{2t};$$

whence it is evident that $c_2 = c_1$. Therefore we have

$$x = ce^{\frac{5}{2}t} + \tfrac{2}{5}\,e^{5t} - 2\,e^{2t},$$
$$y = ce^{\frac{5}{2}t} - \tfrac{3}{5}\,e^{5t} - e^{2t},$$

as the solutions of the given equations.

109. Equations of the second order. Equations of the second order are of special importance in applications. It is of interest, therefore, to sketch methods by which such equations can sometimes be solved.

In the first place, it is sometimes possible to reduce such an equation to one of the first order by placing

$$\frac{dy}{dx} = p. \tag{1}$$

We may then place either

$$\frac{d^2y}{dx^2}=\frac{dp}{dx}, \tag{2}$$

or, since

$$\frac{dp}{dx}=\frac{dp}{dy}\frac{dy}{dx},$$

$$\frac{d^2y}{dx^2}=p\frac{dp}{dy}. \tag{3}$$

It is clear that if the given equation is of the type

$$f\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0, \tag{4}$$

that is, if it does not contain y explicitly, then the substitutions (1) and (2) give an equation in x and $\frac{dp}{dx}$; whereas if the given equation is of the type

$$f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0, \tag{5}$$

that is, if it does not contain x explicitly, substitutions (1) and (3) give an equation in y and $\frac{dp}{dy}$. If p is found in either case, equation (1) then gives an equation of the first order to determine y.

As a particular example consider the case of a particle so moving in a straight line that the force acting on it is a function of the displacement from a fixed origin. Then if s is the distance from the origin, and t is time, the differential equation of the motion is

$$\frac{d^2s}{dt^2}=f(s).$$

Placing

$$\frac{ds}{dt}=p, \qquad \frac{d^2s}{dt^2}=p\frac{dp}{ds},$$

we have

$$p\,dp=f(s)ds;$$

whence

$$p^2=2\int f(s)ds+c_1,$$

and, finally,

$$\int\frac{ds}{\sqrt{2\int f(s)ds+c_1}}=t+c_2.$$

In carrying out this solution it will usually be desirable or even necessary to determine the constant c_1 before integrating.

When the differential equation of the second order is of the linear type, other methods may be applicable. Consider the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R, \tag{6}$$

where P, Q, and R are independent of y. We shall enumerate cases where the solution is possible.

CASE I. If P and Q are constants, the equation is to be solved by the methods of §§ 102–106.

CASE II. If P and Q are not constants, the equation is possibly of the type discussed in § 107.

CASE III. It is possible that the left-hand member of (6) may be an exact differential. This happens when $Q = \frac{dP}{dx}$, and then equation (6) may be written

$$\frac{d}{dx}\left(\frac{dy}{dx} + Py\right) = R;$$

whence

$$\frac{dy}{dx} + Py = \int R\,dx + c_1.$$

This is the linear type of Case IV, § 91.

CASE IV. It is sometimes true that a solution of the reduced equation

$$\frac{d^2y}{dx} + P\frac{dy}{dx} + Qy = 0 \tag{7}$$

is known or may be found by inspection. Let $y = y_1(z)$ be such a solution, and substitute

$$y = y_1 z$$

in (6). We have

$$y_1\frac{d^2z}{dx^2} + \left(2\frac{dy_1}{dx} + Py_1\right)\frac{dz}{dx} = R, \tag{8}$$

which is of the type to which substitutions (1) and (2) are applicable.

CASE V. Occasionally two solutions of the reduced equation (7) are known or can be found by inspection.

In that case, if $y = y_1$ and $y = y_2$ are the two known solutions of (7), then

$$y = c_1y_1 + c_2y_2 + I$$

is the general solution of (6), and I may sometimes be found by the method of variation of constants or otherwise.

CASE VI. We may try placing

$$y = uz$$

in (6). There results

$$u\frac{d^2z}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dz}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)z = R. \qquad (9)$$

If u can be so chosen that the coefficient of z vanishes, we have again equation (8), already discussed. Otherwise let us take u so that the coefficient of $\frac{dz}{dx}$ shall vanish; that is, we take

$$u = e^{-\frac{1}{2}\int P\,dx}.$$

Then, after a few reductions, equation (9) becomes

$$\frac{d^2z}{dx^2} + \left(Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2\right)z = Re^{\frac{1}{2}\int P\,dx}. \qquad (10)$$

Therefore, if $Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$ is equal to a constant, equation (10) is of the type of § 104; if $Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = \frac{k^2}{x^2}$, where k^2 is a constant, equation (10) is of the type of § 107.

CASE VII. Finally we may try the effect of changing the independent variables from x to t, where t is a function of x to be determined.

Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = t'\frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = t'^2\frac{d^2y}{dt} + t''\frac{dy}{dt},$$

and the differential equation (6) becomes

$$\frac{d^2y}{dt^2} + \frac{t'' + Pt'}{t'^2}\frac{dy}{dt} + \frac{Q}{t'^2}y = \frac{R}{t'^2}. \qquad (11)$$

We wish to choose t so that (11) is of the type of § 104. We accordingly choose t so that

$$t'^2 = kQ,$$

where k may be ± 1 or any other conveniently chosen constant. Then if $\frac{t'' + Pt'}{t'^2}$ becomes a constant, the desired result is obtained. We may also endeavor to reduce (11) to the type of § 107, but nothing new is thus obtained.

It should be said that all these methods of solving a differential equation are likely to fail with equations which occur in practice. In such cases recourse may be had to a series expansion. The differential equation, or the series arising from it, then often defines a new function, which must be studied. We shall illustrate this method in the next section and study the Bessel functions from this standpoint in the next chapter.

110. Legendre's equation. Consider the equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0, \tag{1}$$

where n is a constant. To solve it we shall assume the series

$$y=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\cdots \tag{2}$$

and endeavor to determine the first exponent m and the coefficients a_i so as to satisfy (1).

Computing $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (2) and forming the terms of (1), we have

$$\begin{aligned}\frac{d^2y}{dx^2} &= m(m-1)a_0x^{m-2}+(m+1)ma_1x^{m-1}\\ &\qquad\qquad+(m+2)(m+1)a_2x^m+\cdots,\\ -x^2\frac{d^2y}{dx^2} &= \qquad\qquad -m(m-1)a_0x^m-\cdots,\\ -2x\frac{dy}{dx} &= \qquad\qquad -2ma_0x^m-\cdots,\\ n(n+1)y &= \qquad\qquad n(n+1)a_0x^m+\cdots.\end{aligned} \tag{3}$$

The sum of these terms must be identically zero if (1) is to be satisfied. Hence we must have, in the first place,

$$m(m-1)a_0=0, \tag{4}$$

$$(m+1)ma_1=0, \tag{5}$$

$$(m+2)(m+1)a_2-(m-n)(m+n+1)a_0=0. \tag{6}$$

Equation (4) gives either $m=0$ or $m=1$, with a_0 arbitrary in either case. Let us take the first case, $m=0$. Then from (5) a_1 is arbitrary, and from (6)

$$a_2=-\frac{n(n+1)}{2}a_0. \tag{7}$$

Subsequent coefficients may be obtained by taking more terms of the series (3), but it will be better to obtain the general law of the coefficients. To do this we will find the term containing x^{m+r-2} in each of the series (3), this term being chosen because it contains a_r in the expansion of $\dfrac{d^2y}{dx^2}$. We have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \cdots + (m+r)(m+r-1)a_r x^{m+r-2} + \cdots,\\
-x^2\frac{d^2y}{dx^2} &= \cdots - (m+r-2)(m+r-3)a_{r-2}x^{m+r-2} - \cdots,\\
-2x\frac{dy}{dx} &= \cdots - 2(m+r-2)a_{r-2}x^{m+r-2} - \cdots,\\
n(n+1)y &= \cdots + n(n+1)a_{r-2}x^{m+r-2} + \cdots,
\end{aligned}$$

and since the sum of these terms must vanish, we have, after combining and factoring the coefficients of a_{r-2},

$$(m+r)(m+r-1)a_r + (n-m-r+2)(n+m+r-1)a_{r-2} = 0.$$

We are considering the case in which $m = 0$, so that we have

$$a_r = -\frac{(n-r+2)(n+r-1)}{r(r-1)}a_{r-2}, \tag{8}$$

which enables us to determine any coefficient from the one which precedes it by two terms. We have, accordingly,

$$\begin{aligned}
y = a_0\left(1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right)\\
+ a_1\Bigg(x - \frac{(n-1)(n+2)}{3!}x^3\\
+ \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots\Bigg).
\end{aligned} \tag{9}$$

It is easily shown by the ratio test that each of these series converges in the interval $(-1, +1)$.

Since a_0 and a_1 are arbitrary, this is the general solution of (1) by theorem I, § 102. The student may verify the fact that if he considers the assumption $m = 1$, discarded in solving (4), he gets nothing new, but only the second series in (9).

By taking either a_0 or a_1 equal to zero, solution (9) becomes a single series, and particular interest attaches to the cases in which this series reduces to a polynomial. This evidently happens to the first series when n is an even integer and to the second series when

n is an odd integer. By giving to the coefficient a_0 or a_1, as the case may be, such a numerical value that the polynomial becomes equal to unity when x is unity, we obtain the following system of polynomials:

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= x,\\
P_2(x) &= \tfrac{3}{2}x^2 - \tfrac{1}{2},\\
P_3(x) &= \tfrac{5}{2}x^3 - \tfrac{3}{2}x,\\
P_4(x) &= \frac{7\cdot 5}{4\cdot 2}x^4 - 2\,\frac{5\cdot 3}{4\cdot 2}x^2 + \frac{3\cdot 1}{4\cdot 2},\\
P_5(x) &= \frac{9\cdot 7}{4\cdot 2}x^5 - 2\,\frac{7\cdot 5}{4\cdot 2}x^3 + \frac{5\cdot 3}{4\cdot 2}x.
\end{aligned}$$

These are called the Legendre polynomials. Each satisfies a Legendre differential equation in which n has the value indicated by the subscript.

The Legendre polynomial $P_n(x)$ is the coefficient of h^n in the expansion of

$$\phi = (1 - 2\,xh + h^2)^{-\frac{1}{2}} \tag{10}$$

in ascending powers of h. The student may easily verify this for the lower values of n by actually expanding (10) by the binomial theorem. To prove it for the general term, we first form the following identities by differentiating (10):

$$(1 - 2\,hx + h^2)\frac{\partial\phi}{\partial h} = (x - h)\phi, \tag{11}$$

$$h\frac{\partial\phi}{\partial h} = (x - h)\frac{\partial\phi}{\partial x}. \tag{12}$$

Now place

$$\phi = \sum_{n=0}^{n=\infty} A_n h^n. \tag{13}$$

It is obvious that A_n is a polynomial in x of degree n. Also, if $x = 1$ in (10), then $\phi = \frac{1}{1-h}$, and therefore A_n in (13) is equal to 1 when $x = 1$. Hence if we can show that A_n satisfies Legendre's equation, it will be identified with $P_n(x)$, since our solution of Legendre's equation has given us the only polynomials which satisfy that equation and have the value 1 when $x = 1$.

Substitute from (13) in (11) and (12) and equate coefficients of h^{n-1} on both sides of the resulting equations. We get

$$nA_n - (2n-1)xA_{n-1} + (n-1)A_{n-2} = 0, \tag{14}$$

$$x\frac{dA_{n-1}}{dx} - \frac{dA_{n-2}}{dx} = (n-1)A_{n-1}. \tag{15}$$

Replacing n by $n+1$ in (15), we have the equivalent formula,

$$x\frac{dA_n}{dx} - \frac{dA_{n-1}}{dx} = nA_n. \tag{16}$$

By differentiating (14) with respect to x and eliminating $\frac{dA_{n-2}}{dx}$ by (15), we have

$$\frac{dA_n}{dx} - x\frac{dA_{n-1}}{dx} = nA_{n-1}. \tag{17}$$

Then if (16) is multiplied by $-x$ and added to (17), we have

$$(1-x^2)\frac{dA_n}{dx} = n(A_{n-1} - xA_n). \tag{18}$$

By differentiating (18) with respect to x and simplifying the result by means of (16), we have, finally,

$$(1-x^2)\frac{d^2A_n}{dx^2} - 2x\frac{dA_n}{dx} + n(n+1)A_n = 0. \tag{19}$$

This shows that A_n is a solution of Legendre's equation. Hence for the reasons already stated $A_n(x)$ is the same as $P_n(x)$. Formulas (14) to (18) may be rewritten, replacing A_n by P_n, and give important relations connecting Legendre polynomials with different indices.

Another class of polynomials related to the Legendre polynomials may be found as follows:

Differentiate equation (1) m times with respect to x and place $\frac{d^my}{dx^m} = v$. We obtain

$$(1-x^2)\frac{d^2v}{dx^2} - 2x(m+1)\frac{dv}{dx} + (n-m)(n+m+1)v = 0, \tag{20}$$

an equation which is satisfied by

$$v = \frac{d^mP_n(x)}{dx^m}. \tag{21}$$

In (20) place $w = v(1 - x^2)^{\frac{m}{2}}$. We get

$$(1 - x^2)\frac{d^2w}{dx^2} - 2x\frac{dw}{dx} + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]w = 0. \qquad (22)$$

This equation, which differs from Legendre's equation in an added term involving m, is called the associated Legendre equation. It is satisfied by

$$w = (1 - x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x). \qquad (23)$$

This value of w is the associated Legendre polynomial and is denoted by $P_n{}^m(x)$. If $m > n$, $P_n{}^m(x) = 0$.

EXERCISES

1. Assuming that the solution of $\frac{d^2y}{dx^2} + y = 0$ is $y = c_1 \sin x + c_2 \cos x$, find by the method of variation of constants the solution of $\frac{d^2y}{dx^2} + y = \tan x$.

2. Assuming that the solution of $\frac{d^2y}{dx^2} + y = 0$ is $y = c_1 \sin x + c_2 \cos x$, find by the method of variation of constants the solution of $\frac{d^2y}{dx^2} + y = \sec x$.

3. Assuming that the solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ is $y = (c_1 + c_2x)e^x$, find by the method of variation of constants the solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{1 - x}.$$

4. Assuming that the solution of $(1 - x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ is

$$y = c_1x + c_2e^x,$$

find by the method of variation of constants the solution of

$$(1 - x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1 - x)^2.$$

Solve the following equations:

5. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 4 \cos 2x$.

6. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 5$.

7. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3x^2 + 1$.

8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} + 5e^{2x}$.

9. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = x^2 + 1$.

10. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{3x} \sin 2x$.

11. $\dfrac{d^2y}{dx^2} + 9y = \cos 3x.$

12. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 2 + e^{3x}.$

13. $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 3.$

14. $\dfrac{d^3y}{dx^3} + y = \cos x.$

15. $\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = e^{2x} + 1.$

16. $x^2\dfrac{d^2y}{dx^2} - 6x\dfrac{dy}{dx} + 6y = x^2.$

17. $x^2\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 6y = x\log x.$

18. $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{dy}{dx} + x\dfrac{dy}{dx} = x^2.$

19. $\dfrac{dx}{dt} + \dfrac{dy}{dt} + y = 0,$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x + 3y = 0.$

20. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x = e^{-t},$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2y = \cos 2t.$

21. $\dfrac{d^2x}{dt^2} - \dfrac{dy}{dt} = t,$

$\dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} + 2x - y = t^2.$

22. $\dfrac{d^2y}{dx^2}\dfrac{dy}{dx} + 2x = 0.$

23. $x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 0.$

24. $(a - x)\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}.$

25. $(y + a)\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0.$

26. $y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 + \dfrac{dy}{dx} = 0.$

27. $\dfrac{d^2y}{dx^2} + 2y\dfrac{dy}{dx} = 0.$

28. $y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2 + 1.$

29. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + (2 + 4x^2)y = x^3e^{2x}.$

30. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + xy = \sin 2x.$

31. $x^2\dfrac{d^2y}{dx^2} - 2x^2\dfrac{dy}{dx} + (x^2 - 6)y = 0.$

32. $\dfrac{d^2y}{dx^2} + 2\tan x\dfrac{dy}{dx} + 2y\tan^2 x = \sin 2x.$

33. $\dfrac{d^2y}{dx^2} + (3e^x - 1)\dfrac{dy}{dx} + 2e^{2x}y = 2e^{2x}.$

34. $\dfrac{d^2y}{dx^2} + (2\sin x - \operatorname{ctn} x)\dfrac{dy}{dx} + y\sin^2 x = \sin^4 x.$

Solve the following equations by expansion into series:

35. $\dfrac{d^2y}{dx^2} + k^2y = 0.$

36. $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - ny = 0.$

37. $\dfrac{d^2y}{dx^2} + nxy = 0.$

38. $x^2\dfrac{d^2y}{dx^2} + x^2\dfrac{dy}{dx} + (x - 2)y = 0.$

39. $(x - x^2)\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0.$

40. Prove that $\dfrac{dP_{n+1}}{dx} - \dfrac{dP_{n-1}}{dx} = (2n+1)P_n.$

41. Verify for small values of n the general formula

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

42. A particle moves in a straight line from a distance a toward a center of force which attracts with a magnitude equal to $\dfrac{k}{r^3}\cdot$ If the particle was originally at rest, how long will it be before it reaches the center?

43. A particle moves in a straight line from a distance a toward a center of force which attracts with a magnitude equal to $kr^{-\frac{5}{3}}$. If the particle was originally at rest, how long will it be before it reaches the center?

44. A particle begins to move from a distance a toward a fixed center of force which repels with a magnitude equal to k times the distance of the particle from the center. If its initial velocity is $\sqrt{ka^2}$, show that the particle will continually approach the center but never reach it.

45. A particle moves along a straight line toward a center of force which attracts directly as the distance from the center. If it starts from a position of rest a units from the center, what velocity will it have acquired when it has traversed half the distance to the center?

46. A particle moves in a straight line from a distance a toward a center of force which attracts with a magnitude equal to $\dfrac{1}{2r^2}$, r denoting the distance of the particle from the center. If the particle had an initial velocity of $\dfrac{1}{\sqrt{a}}$, how long will it take to traverse half the distance to the center?

47. A particle of unit mass moving in a straight line is acted on by an attracting force in its line of motion directed toward a center and proportional to the distance of the particle from the center, and also by a periodic force equal to $a \cos kt$. Determine its motion.

48. A particle of unit mass moving in a straight line is acted on by three forces: an attracting force in its line of motion directed toward a center and proportional to the distance of the particle from the center, a resisting force proportional to the velocity of the particle, and a periodic force equal to $a \cos kt$. Determine the motion of the particle.

49. Under what conditions will the oscillations of the particle in Ex. 48 become very large as the time increases?

CHAPTER XII

BESSEL FUNCTIONS

111. Bessel's differential equation. Consider the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0, \tag{1}$$

where n is a constant. To solve, we proceed as in § 110, assuming for y the series

$$y=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\cdots+a_rx^{m+r}+\cdots. \tag{2}$$

We get

$$\begin{aligned}
x^2\frac{d^2y}{dx^2}&=\quad m(m-1)a_0x^m+(m+1)ma_1x^{m+1}\\
&\qquad\qquad+(m+2)(m+1)a_2x^{m+2}+\cdots,\\
x\frac{dy}{dx}&=ma_0x^m+(m+1)a_1x^{m+1}+(m+2)a_2x^{m+2}+\cdots,\\
-n^2y&=\qquad -n^2a_0x^m-n^2a_1x^{m+1}-n^2a_2x^{m+2}-\cdots,\\
x^2y&=\qquad\qquad\qquad\qquad a_0x^{m+2}+\cdots.
\end{aligned}$$

Equating to zero the sum of the coefficients of each of the first three powers of x, we have

$$(m^2-n^2)a_0=0, \tag{3}$$

$$[(m+1)^2-n^2]a_1=0, \tag{4}$$

$$[(m+2)^2-n^2]a_2+a_0=0. \tag{5}$$

To obtain the general expression for the coefficients, we have

$$\begin{aligned}
x^2\frac{d^2y}{dx^2}&=\cdots+(m+r)(m+r-1)a_rx^{m+r}+\cdots,\\
x\frac{dy}{dx}&=\qquad\cdots+(m+r)a_rx^{m+r}+\cdots,\\
-n^2y&=\qquad\quad\cdots-n^2a_rx^{m+r}-\cdots,\\
x^2y&=\qquad\quad\cdots+a_{r-2}x^{m+r}+\cdots.
\end{aligned}$$

Equating to zero the sum of these coefficients, we have

$$[(m+r)^2-n^2]a_r+a_{r-2}=0. \tag{6}$$

Equation (3) may be satisfied by $m = \pm n$. We shall first take $m = n$. Then from (4), (5), and (6) we have

$$a_1 = 0,$$

$$a_2 = -\frac{a_0}{2(2n+2)},$$

$$a_r = -\frac{a_{r-2}}{r(2n+r)}.$$

By use of these results we obtain the series

$$y_1 = a_0x^n\left(1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \frac{x^6}{2\cdot 4\cdot 6(2n+2)(2n+4)(2n+6)} + \cdots\right). \qquad (7)$$

Similarly, placing $m = -n$, we obtain the series

$$y_2 = a_0x^{-n}\left(1 + \frac{x^2}{2(2n-2)} + \frac{x^4}{2\cdot 4(2n-2)(2n-4)} + \frac{x^6}{2\cdot 4\cdot 6(2n-2)(2n-4)(2n-6)} + \cdots\right). \qquad (8)$$

If $n = 0$, the two series (7) and (8) are identical. If n is a positive integer, series (8) is meaningless, since some of the coefficients become infinite. If n is a negative integer, series (7) is meaningless, since some of the coefficients become infinite. Hence if n is zero or an integer, we have in (7) and (8) only one particular solution of the differential equation, and another particular solution must be found before the general solution is known. But if n is not zero or an integer, each of the two series converges for all values of x, as the student may easily show by means of the ratio test (§ 19). Hence we have two particular solutions of (1), and the general solution is

$$y = c_1y_1 + c_2y_2.$$

112. Bessel functions of integral order. We shall restrict ourselves in this section to the case in which n is an integer, leaving to another place the consideration of fractional values of n. We consider in the first place that n is a positive integer. The series for y_1 in § 111 converges. To make the solution definite we place

$$a_0 = \frac{1}{2^n n!}$$

and have the Bessel function of the first kind of order n, where n is a positive integer. This is denoted by $J_n(x)$, and accordingly

$$J_n(x) = \frac{x^n}{2^n n!}\left(1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4 2!(n+1)(n+2)} - \frac{x^6}{2^6 3!(n+1)(n+2)(n+3)} + \cdots\right)$$

$$= \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}(n+1)!} + \frac{x^{n+4}}{2^{n+4}2!(n+2)!} - \frac{x^{n+6}}{2^{n+6}3!(n+3)!} + \cdots.$$

The general term is

$$(-1)^k \frac{x^{n+2k}}{2^{n+2k}k!(n+k)!}.$$

If we place $k = 0$ we get the first term of the expansion of $J_n(x)$ provided we place $0! = 1$, which is customary, as an obvious extension from the general relation $(n-1)! = \frac{n!}{n}$.

We have, finally,

$$J_n(x) = \sum_{k=0}^{k=\infty}(-1)^k \frac{x^{n+2k}}{2^{n+2k}k!(n+k)!}. \tag{1}$$

This holds for positive integral n's and also for $n = 0$. In particular,

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \cdots + (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2} + \cdots, \tag{2}$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 2!} + \frac{x^5}{2^5 2!3!} - \cdots + (-1)^k \frac{x^{2k+1}}{2^{2k+1}k!(k+1)!} + \cdots. \tag{3}$$

From the series given above many important relations may be obtained. We shall first prove the relation

$$\frac{d}{dx} J_0(x) = - J_1(x). \tag{4}$$

It is evident that the derivative of the second term of (2) is the negative of the first term of (3), and that the derivative of the third term of (2) is the negative of the second term of (3). To show the general law we take from (2) the term next after the one written; namely,

$$(-1)^{k+1} \frac{x^{2k+2}}{2^{2k+2}[(k+1)!]^2}.$$

Its derivative is $(-1)^{k+1}\dfrac{(2k+2)x^{2k+1}}{2^{2k+2}[(k+1)!]^2}$,

which reduces to the last term written in (3).

Hence equation (4) follows.

The general term of $x^nJ_n(x)$ is

$$(-1)^k\frac{x^{2n+2k}}{2^{n+2k}k!(n+k)!},$$

and if this is differentiated with respect to x, it becomes

$$(-1)^k\frac{x^{2n+2k-1}}{2^{n+2k-1}k!(n+k-1)!},$$

which is the general term of $J_{n-1}(x)$ multiplied by x^n. Hence

$$\frac{d}{dx}[x^nJ_n(x)]=x^nJ_{n-1}(x). \tag{5}$$

In the same manner we may prove that

$$\frac{d}{dx}[x^{-n}J_n(x)]=-x^{-n}J_{n+1}(x). \tag{6}$$

From (5) we have

$$nx^{n-1}J_n(x)+x^n\frac{dJ_n(x)}{dx}=x^nJ_{n-1}(x),$$

or, rearranged,
$$\frac{dJ_n(x)}{dx}=J_{n-1}(x)-\frac{n}{x}J_n(x). \tag{7}$$

Similarly, from (6),

$$\frac{dJ_n(x)}{dx}=\frac{n}{x}J_n(x)-J_{n+1}(x). \tag{8}$$

Then by combining (7) and (8) we have

$$J_{n-1}(x)-J_{n+1}(x)=2\frac{dJ_n(x)}{dx}, \tag{9}$$

$$J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x). \tag{10}$$

We have as yet no definition of $J_n(x)$ when n is a negative integer. Series (7), § 111, then fails to converge and series (8), § 111, gives nothing new, for if we place $n=-m$ in (8) we have simply series (7) with $n=m$. We shall therefore introduce a definition for $J_{-n}(x)$ with n an integer by demanding that the relation (10) shall be true for negative values of n.

If we place $n=0$ in (10), we have

$$J_{-1}(x) = -J_1(x). \tag{11}$$

If we place $n=-1$ in (10), we have

$$J_{-2}(x) + J_0(x) = -\frac{2}{x} J_{-1}(x),$$

whereas if we place $n=1$ in (10), we have

$$J_0(x) + J_2(x) = \frac{2}{x} J_1(x).$$

Combining these last two equations with the aid of (11), we have

$$J_{-2}(x) = J_2(x). \tag{12}$$

Again, placing in succession $n=-2$ and $n=2$ in (10), we have

$$J_{-3}(x) + J_{-1}(x) = -\frac{4}{x} J_{-2}(x),$$

$$J_1(x) + J_3(x) = \frac{4}{x} J_2(x);$$

whence, by aid of (11) and (12),

$$J_{-3}(x) = -J_3(x). \tag{13}$$

Continuing in this way we reach the general result,

$$J_{-n}(x) = (-1)^n J_n(x). \tag{14}$$

We have not, however, an essential new solution of equation (1), § 111, and theorem I of § 102 is not yet applicable.

It is now easy to show that equations (5) to (9) are valid when n is a negative integer.

113. Roots of Bessel functions of integral order. We shall prove the following theorem:

Between any two consecutive real roots of $J_n(x)=0$ *lies one and only one real root of* $J_{n+1}(x)=0$.

From Rolle's theorem we know that between two consecutive real roots of $x^{-n}J_n(x)=0$ lies at least one root of the derivative of $x^{-n}J_n(x)$, which, by (6), § 112, is $-x^{-n}J_{n+1}(x)$. No root of $x^{-n}J_n(x)$ can coincide with that of its derivative, $-x^{-n}J_{n+1}(x)$, for if we place $y=x^{-n}J_n(x)$ we have

$$\frac{d^2y}{dx^2} + \frac{1+2n}{x}\frac{dy}{dx} + y = 0;$$

and if both y and $\frac{dy}{dx}$ are zero for the same value of x, so also $\frac{d^2y}{dx^2}=0$. Then all derivatives, computed from the differential equation, are zero, and y is a constant.

Again, if in (5), § 112, we replace n by $n+1$, we infer from Rolle's theorem that between any two roots of $x^{n+1}J_{n+1}(x)=0$ lies at least one root of $x^{n+1}J_n(x)=0$.

Hence, disregarding for the moment the case in which a root of $x^{-n}J_n(x)=0$ is zero, it follows that the roots of $J_n(x)=0$ and $J_{n+1}(x)=0$ lie as stated in the theorem.

To cover the case in which $x=0$, let $x=\xi$ be the smallest positive root of $J_n(x)=0$. Then $x=-\xi$ is also a root of $J_n(x)=0$, as is seen by the series expansion. Hence by Rolle's theorem and (6), §112, there is at least one root of $x^{-n}J_{n+1}(x)=0$ between $x=\xi$ and $x=-\xi$. Such a root is $x=0$, as is seen from the series expansion for $J_{n+1}(x)$. There can be no other root between ξ and $-\xi$, for if there were, $x=\xi$ would not be the smallest root of $J_n(x)=0$. Hence the theorem is proved.

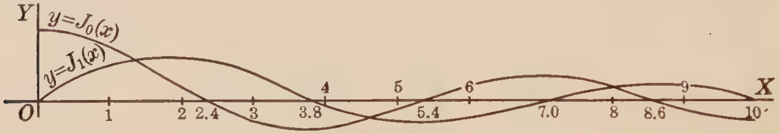

FIG. 95

The theorem is illustrated in the graphs of $y=J_0(x)$ and $y=J_1(x)$ (Fig. 95).

114. Bessel functions of integral order as definite integrals. We shall first prove that the coefficient of t^n in the expansion of $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ is $J_n(x)$, where n is a positive integer. We have, by use of Maclaurin's series,

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=e^{\frac{x}{2}t}\cdot e^{-\frac{x}{2}\frac{1}{t}}$$

$$=\left(1+\frac{x}{2}t+\cdots+\frac{x^l}{2^l}\frac{1}{l!}t^l+\cdots\right)\left(1-\frac{x}{2}\frac{1}{t}+\cdots+(-1)^k\frac{x^k}{2^k}\frac{1}{k!}\frac{1}{t^k}+\cdots\right).$$

We shall obtain the term containing t^n in the product, when n is positive, by taking $l=n+k$ in any term of the first series, multiplying that term by the term in the second series which involves $\frac{1}{t^k}$, and summing on k. We have

$$\sum(-1)^k\frac{x^{n+2k}}{2^{n+2k}k!(n+k)!}t^n=J_n(x)t^n.$$

To find the term containing $\frac{1}{t^n}$ we place $k = l + n$ in any term of the second series, multiply that term by the term of the first series which involves t^l, and sum on l. We get

$$(-1)^n J_n(x)\frac{1}{t^n} = J_{-n}(x)t^{-n}.$$

This is what we wished to show.

Hence

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + J_1(x)t + J_2(x)t^2 + \cdots + J_n(x)t^n + \cdots$$
$$+ J_{-1}(x)t^{-1} + J_{-2}(x)t^{-2} + \cdots + J_{-n}(x)t^{-n} + \cdots. \qquad (1)$$

Now place

$$t = e^{i\phi}, \quad \frac{1}{t} = e^{-i\phi}, \quad t - \frac{1}{t} = e^{i\phi} - e^{-i\phi} = 2\,i \sin \phi. \qquad (2)$$

Then on the left of equation (1) we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{xi \sin \phi} = \cos\,(x \sin \phi) + i \sin\,(x \sin \phi). \qquad (3)$$

On the right we combine each pair of terms containing t^n and t^{-n} and, by use of (11), § 112, we have

$$J_0(x) + J_1(x)\left(t - \frac{1}{t}\right) + J_2(x)\left(t^2 + \frac{1}{t^2}\right) + J_3(x)\left(t^3 - \frac{1}{t^3}\right) + \cdots$$
$$+ J_{2k}(x)\left(t^{2k} + \frac{1}{t^{2k}}\right) + J_{2k+1}(x)\left(t^{2k+1} - \frac{1}{t^{2k+1}}\right) + \cdots. \quad (4)$$

By (2), $$t^{2k} + \frac{1}{t^{2k}} = e^{2ki\phi} + e^{-2ki\phi} = 2 \cos 2\,k\phi,$$

$$t^{2k+1} - \frac{1}{t^{2k+1}} = e^{(2k+1)i\phi} - e^{-(2k+1)i\phi} = 2\,i \sin\,(2\,k+1)\phi,$$

so that (4) becomes

$$J_0(x) + 2\,J_2(x) \cos 2\,\phi + 2\,J_4(x) \cos 4\,\phi + \cdots$$
$$+ 2\,J_{2k}(x) \cos 2\,k\phi + \cdots$$
$$+ i[2\,J_1(x) \sin \phi + 2\,J_3(x) \sin 3\,\phi + \cdots$$
$$+ 2\,J_{2k+1}(x) \sin\,(2\,k+1)\phi + \cdots]. \qquad (5)$$

This is to be placed equal to (3), and the real and imaginary parts equated. The result is

$$\cos\,(x \sin \phi) = J_0(x) + 2\,J_2(x) \cos 2\,\phi + 2\,J_4(x) \cos 4\,\phi + \cdots, \quad (6)$$

$$\sin\,(x \sin \phi) = 2\,J_1(x) \sin \phi + 2\,J_3(x) \sin 3\,\phi$$
$$+ 2\,J_5(x) \sin 5\,\phi + \cdots. \qquad (7)$$

We know from elementary integration that

$$\int_0^\pi \cos^2 n\phi\, d\phi = \int_0^\pi \sin^2 n\phi\, d\phi = \frac{\pi}{2},$$

$$\int_0^\pi \cos k\phi \cos n\phi\, d\phi = \int_0^\pi \sin k\phi \sin n\phi = 0. \quad (k \neq n)$$

Hence if we multiply all terms of (6) by $\cos n\phi\, d\phi$ and all terms of (7) by $\sin n\phi\, d\phi$, and integrate between 0 and π, we have

$$\left.\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^\pi \cos(x \sin\phi)\cos n\phi\, d\phi, &&(n \text{ even or } 0)\\ 0 &= \frac{1}{\pi}\int_0^\pi \cos(x \sin\phi)\cos n\phi\, d\phi, &&(n \text{ odd})\\ J_n(x) &= \frac{1}{\pi}\int_0^\pi \sin(x \sin\phi)\sin n\phi\, d\phi, &&(n \text{ odd})\\ 0 &= \frac{1}{\pi}\int_0^\pi \sin(x \sin\phi)\sin n\phi\, d\phi. &&(n \text{ even}) \end{aligned}\right\} \quad (8)$$

From this it appears that

$$J_n(x) = \frac{1}{\pi}\int_0^\pi [\cos(x\sin\phi)\cos n\phi + \sin(x\sin\phi)\sin n\phi]d\phi$$

whether n be odd or even, and therefore

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - x\sin\phi)d\phi. \quad (n \text{ any integer}) \qquad (9)$$

Another integral form may be obtained as follows:

We have seen (§ 68) that

$$\int_0^\pi \sin^{2n}\phi \cos^{2k}\phi\, d\phi = 2\int_0^{\frac{\pi}{2}} \sin^{2n}\phi\cos^{2k}\phi\, d\phi = \mathrm{B}(n+\tfrac{1}{2}, k+\tfrac{1}{2})$$

$$= \frac{\Gamma(n+\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(n+k+1)}$$

$$= \frac{1\cdot 3\cdots(2n-1)\cdot 1\cdot 3\cdots(2k-1)}{2^{n+k}(n+k)!}\pi.$$

Therefore

$$\frac{1}{(n+k)!} = \frac{2^{n+k}}{1\cdot 3\cdots(2n-1)\cdot 1\cdot 3\cdots(2k-1)\pi}\int_0^\pi \sin^{2n}\phi\cos^{2k}\phi\, d\phi.$$

If we substitute this in the general term of $J_n(x)$, that term becomes

$$(-1)^k \frac{x^{n+2k}}{2^k k! \cdot 1 \cdot 3 \cdots (2n-1) \cdot 1 \cdot 3 \cdots (2k-1)\pi} \int_0^\pi \sin^{2n}\phi \cos^{2k}\phi \, d\phi$$

$$= (-1)^k \frac{x^{n+2k}}{(2k)! \cdot 1 \cdot 3 \cdots (2n-1)\pi} \int_0^\pi \sin^{2n}\phi \cos^{2k}\phi \, d\phi$$

$$= \frac{x^n}{1 \cdot 3 \cdots (2n-1)\pi} \int_0^\pi \sin^{2n}\phi \left[(-1)^k \frac{(x \cos\phi)^{2k}}{(2k)!} \right] d\phi.$$

Let this now be summed on k. Since the Maclaurin expansion for $\cos x$ gives

$$\cos(x \cos\phi) = \sum (-1)^k \frac{(x\cos\phi)^{2k}}{(2k)!},$$

we have

$$J_n(x) = \frac{x^n}{1 \cdot 3 \cdots (2n-1)\pi} \int_0^\pi \sin^{2n}\phi \cos(x\cos\phi) d\phi. \quad (10)$$

Still another integral may be obtained as follows:

In (10) place $t = \cos\phi$.

We have

$$J_n(x) = \frac{x^n}{1 \cdot 3 \cdots (2n-1)\pi} \int_{-1}^{+1} (1-t^2)^{n-\frac{1}{2}} \cos(xt) dt.$$

Evidently $$\int_{-1}^{+1} (1-t^2)^{n-\frac{1}{2}} \sin(xt) dt = 0,$$

since the function to be integrated is an odd function. Therefore

$$J_n(x) = \frac{x^n}{1 \cdot 3 \cdots (2n-1)\pi} \int_{-1}^{+1} (1-t^2)^{n-\frac{1}{2}} [\cos(xt) + i \sin(xt)] dt$$

$$= \frac{x^n}{1 \cdot 3 \cdots (2n-1)\pi} \int_{-1}^{+1} e^{ixt} (1-t^2)^{n-\frac{1}{2}} dt. \quad (11)$$

This is sometimes taken as the starting-point for the discussion of the Bessel functions.

115. The function $I_n(x)$. Another Bessel function of importance is defined as follows:

$$I_n(x) = i^{-n} J_n(ix)$$

$$= i^{-n} \sum \frac{(-1)^k (ix)^{n+2k}}{2^{n+2k} k!(n+k)!}$$

$$= \sum \frac{x^{n+2k}}{2^{n+2k} k!(n+k)!}; \quad (1)$$

in particular, $$I_0(x) = \sum \frac{x^{2k}}{2^{2k}(k!)^2}.$$

Formulas (5) to (10) of § 112 lead to corresponding formulas for $I_n(x)$. Consider

$$\begin{aligned}\frac{d}{dx}[x^n I_n(x)] &= \frac{d}{dx}[x^n i^{-n} J_n(ix)]\\ &= \frac{d}{dx}\left[\frac{(ix)^n}{i^{2n}} J_n(ix)\right]\\ &= \frac{i}{i^{2n}}\frac{d}{d(ix)}[(ix)^n J_n(ix)]\\ &= \frac{1}{i^{2n-1}}(ix)^n J_{n-1}(ix) \qquad \text{[by (5), § 112]}\\ &= x^n i^{-n+1} J_{n-1}(ix) = x^n I_{n-1}(x). \qquad (2)\end{aligned}$$

Similarly, $$\frac{d}{dx}[x^{-n}I_n(x)] = x^{-n}I_{n+1}(x). \qquad (3)$$

The two formulas (2) and (3) lead to

$$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x), \qquad (4)$$

and $$I_{n-1}(x) + I_{n+1}(x) = 2\frac{d}{dx} I_n(x). \qquad (5)$$

It is also easy to show that

$$y = I_n(x)$$

satisfies the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+n^2)y = 0. \qquad (6)$$

116. The Bessel function of fractional order. Referring to the definition of $J_n(x)$ for integral n, we may form the definition for fractional n's by replacing $n!$ by $\Gamma(n+1)$. Now both of the series (7) and (8), § 111, converge, and we have the two functions

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\, y_1 = \sum \frac{(-1)^k x^{n+2k}}{2^{n+2k}k!\Gamma(n+k+1)} \qquad (1)$$

and $$J_{-n}(x) = \frac{x^{-n}}{2^{-n}\Gamma(-n+1)}\, y_2 = \sum \frac{(-1)^k x^{-n+2k}}{2^{-n+2k}k!\Gamma(-n+k+1)}. \qquad (2)$$

The complete solution of the Bessel equation is, then,

$$y = c_1 J_n(x) + c_2 J_{-n}(x).$$

The formulas already derived, so far as they depend upon properties of $n!$ which are shared by $\Gamma(n+1)$, evidently hold, but those which are based upon the hypothesis that n is an integer evidently fail. It appears, then, that formulas (5) to (10), § 112, still hold. Formula (9), § 114, is not applicable, but formulas (10) and (11), § 114, may be replaced by

$$J_n(x) = \frac{x^n}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_0^\pi \sin^{2n}\phi \cos(x\cos\phi)d\phi, \qquad (3)$$

$$J_n(x) = \frac{x^n}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_{-1}^1 e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt, \qquad (4)$$

since the proofs of § 114 may be repeated for fractional values of n with the use of the Gamma functions.

Special interest attaches to the case in which $n = \frac{1}{2}$ or an odd multiple of $\frac{1}{2}$.

We have $$J_{\frac{1}{2}}(x) = \sum \frac{(-1)^k x^{\frac{1}{2}} x^{2k}}{2^{\frac{1}{2}} 2^{2k} k!\,\Gamma(k+\frac{3}{2})}.$$

But $$\Gamma(k+\tfrac{3}{2}) = (k+\tfrac{1}{2})(k-\tfrac{1}{2})\cdots\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$
$$= \frac{(2k+1)(2k-1)\cdots 1\sqrt{\pi}}{2^{k+1}}.$$

Therefore $$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sum\frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sqrt{\frac{2}{\pi x}}\sin x$$

by Maclaurin's series for $\sin x$.

In a similar manner, $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$

By use of (10), § 112, $J_n(x)$ may now be found in terms of $\sin x$ and $\cos x$ for any n of the form $\pm\frac{2k+1}{2}$.

117. Bessel functions of the second kind. When n is not an integer, the complete solution of the Bessel equation is

$$y = c_1 J_n(x) + c_2 J_{-n}(x); \qquad (1)$$

but when n is an integer,

$$J_{-n}(x) = (-1)^n J_n(x), \qquad (2)$$

and (1) contains only one arbitrary constant and is therefore not the complete equation. We must therefore seek another particular solution.

We shall apply theorem IV, § 102, and place

$$y = J_n(x)u \qquad (3)$$

in the Bessel equation. Then if $p = \frac{du}{dx}$ and $J_n'(x) = \frac{d}{dx}[J^n(x)]$, the equation for p is readily found to be

$$\left(\frac{2\,J_n'(x)}{J_n(x)} + \frac{1}{x}\right)dx + \frac{dp}{p} = 0; \tag{4}$$

whence
$$2\log J_n(x) + \log x + \log p = c.$$

Therefore
$$p = \frac{C_1}{x[J_n(x)]^2}, \tag{5}$$

and
$$u = C_1 \int \frac{dx}{x[J_n(x)]^2} + C_2. \tag{6}$$

By using the power series for $J_n(x)$ we can write (5) in the form

$$p = \frac{1}{x^{2n+1}}(c_0 + c_1x^2 + c_2x^4 + \cdots), \tag{7}$$

where the coefficients do not need to be explicitly determined for our present purpose. It is essential to notice, however, that the series for p involves a term

$$\frac{c_n}{x} \tag{8}$$

if n is a positive integer, and that the coefficient c_n can be taken as unity because of the arbitrary constant C_1 in (6). Hence, from (7), we have for u a power series of the form

$$u = b_0x^{-2n} + \cdots + b_{n-1}x^{-2} + C_3 + \log x + b_{n+1}x^2 + \cdots. \tag{9}$$

Using this in (3) we get

$$y = J_n(x)\log x + P(x), \tag{10}$$

where $P(x)$ is a power series in x arising from the multiplication of the series (9) by the series for $J_n(x)$. We have then

$$P(x) = a_0x^{-n} + a_2x^{-n+2} + \cdots + a_{2r}x^{-n+2r} + \cdots, \tag{11}$$

and shall proceed to determine the coefficients by substituting the solution (10) in the Bessel equation. We get

$$2\,xJ_n'(x) + x^2P''(x) + xP'(x) + (x^2 - n^2)P(x) = 0, \tag{12}$$

which can be written

$$\sum_{k=0}^{k=\infty} 2(n + 2\,k)A_kx^{n+2k} + \sum_{r=0}^{r=\infty} B_{2r}x^{-n+2r} = 0, \tag{13}$$

where
$$A_k = \frac{(-1)^k}{2^{n+2k}k!(n+k)!}, \tag{14}$$

$$B_0 = 0, \tag{15}$$

$$B_{2r} = 4\,r(r-n)a_{2r} + a_{2r-2}. \quad (r > 0) \tag{16}$$

When $r < n$, the terms of the second series in (13) involve powers of x which do not occur in the first series. Hence for such values of r we have

$$B_{2r} = 0, \quad (r < n) \tag{17}$$

which, from (16), gives the recurring formula

$$a_{2r} = \frac{a_{2r-2}}{4r(n-r)}; \quad (r < n) \tag{18}$$

whence

$$a_{2r} = \frac{a_0}{2^{2r} r!(n-1)(n-2)\cdots(n-r)}, \quad (r < n) \tag{19}$$

where a_0 is still to be determined.

On the other hand, when $r \geqq n$, the terms of the first series in (13) combine with the terms of the second series. We place

$$r = n + k, \quad (k = 0, 1, 2, \cdots \infty) \tag{20}$$

and have as the coefficient of x^{n+2k}

$$2(n + 2k)A_k + B_{2n+2k} = 0,$$

which gives

$$2(n+2k)A_k + 4k(n+k)a_{2n+2k} + a_{2n+2k-2} = 0. \tag{21}$$

When $k = 0$, (21) gives

$$2nA_0 + a_{2n-2} = 0. \tag{22}$$

We determine A_0 by placing $r = n - 1$ in (19) and determine a_{2n-2} by placing $k = 0$ in (14). We then find, from (22), that

$$a_0 = -\frac{(n-1)!}{2^{-n+1}}, \tag{23}$$

so that, from (19),

$$a_{2r} = -\frac{1}{2}\frac{(n-r-1)!}{2^{-n+2r}r!}. \quad (r < n) \tag{24}$$

When $k > 0$, we may write (21) in the form

$$-2\frac{a_{2n+2k}}{A_k} = \frac{a_{2n+2k-2}}{2k(n+k)A_k} + \frac{1}{k} + \frac{1}{n+k}. \tag{25}$$

From (14) we have

$$2k(n+k)A_k = -\tfrac{1}{2}A_{k-1}.$$

Hence (25) gives us the recurring formula

$$-2\frac{a_{2n+2k}}{A_k} = -2\frac{a_{2n+2k-2}}{A_{k-1}} + \frac{1}{k} + \frac{1}{n+k}. \tag{26}$$

Our formulas leave a_{2n} undetermined, since k cannot be placed equal to zero in (26) and r is less than n in (24). Therefore we may give a_{2n} an arbitrary value, which we shall so choose that

$$-2\frac{a_{2n}}{A_0} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}. \tag{27}$$

By repeated application of (26) we have now

$$a_{2n+2k} = -\frac{1}{2}A_k\left[1 + \frac{1}{2} + \cdots + \frac{1}{k} + 1 + \frac{1}{2} + \cdots + \frac{1}{n+k}\right]. \tag{28}$$

All the coefficients of $P(x)$ in (11) have now been determined by formulas (24) and (28). Hence we have from (10) the solution of the Bessel equation which we shall call $K_n(x)$; namely,

$$\begin{aligned} K_n(x) = J_n(x)\log x - \frac{1}{2}\sum_{r=0}^{r=n-1}\frac{(n-r-1)!\,x^{-n+2r}}{2^{-n+2r}r!} \\ - \frac{1}{2}\sum_{k=0}^{k=\infty}\frac{(-1)^k x^{n+2k}}{2^{n+2k}k!\,(n+k)!}\left[1 + \frac{1}{2} + \cdots + \frac{1}{k}\right. \\ \left. + 1 + \frac{1}{2} + \cdots + \frac{1}{n+k}\right]. \end{aligned} \tag{29}$$

In the second summation, when $k = 0$ the terms $1 + \frac{1}{2} + \cdots + \frac{1}{k}$ are to be omitted.

We have now as a complete solution of the Bessel equation

$$y = c_1 J_n(x) + c_2 K_n(x). \tag{30}$$

By giving different values to the arbitrary constants in the last equation various forms of the solution of the second kind may be obtained, and some of these forms, denoted by various letters, have been used by different authors.

It is possible to show that $K_n(x)$ satisfies the relations (5) to (10), § 112, which are satisfied by $J_n(x)$.

EXERCISES

1. Prove formula (6), § 112.

2. Prove that $\quad \dfrac{d}{dx}[x^n J_n(ax)] = ax^{n-1}J_{n-1}(ax).$

3. Show that $\quad \dfrac{d^2y}{dx^2} + \dfrac{1+2n}{x}\dfrac{dy}{dx} + y = 0$

is satisfied by $\quad y = x^{-n}J_n(x).$

4. Show that $$x\frac{d^2y}{dx^2} + (1+n)\frac{dy}{dx} + y = 0$$
is satisfied by $$y = x^{-\frac{n}{2}}J_n(2\sqrt{x}).$$

5. Show that $$\frac{d^2y}{dx^2} + (1-n)\frac{dy}{dx} + y = 0$$
is satisfied by $$y = x^{\frac{n}{2}}J_n(2\sqrt{x}).$$

6. Show that $$\frac{d^2y}{dx^2} + \frac{1-2n}{x}\frac{dy}{dx} + y = 0$$
is satisfied by $$y = x^nJ_n(x).$$

7. Show that $$\frac{d^2y}{dx^2} + \left(a^2 - \frac{n^2-\frac{1}{4}}{x^2}\right)y = 0$$
is satisfied by $$y = \sqrt{x}J_n(ax).$$

8. Prove that
$$4\,J_n''(x) = J_{n-2}(x) - 2\,J_n(x) + J_{n+2}(x).$$

9. Prove that
$$xJ_n(x) = 2(n+1)J_{n+1}(x) - xJ_{n+2}(x).$$

10. Prove that
$$2^3\,J_n'''(x) = J_{n-3}(x) - 3\,J_{n-1}(x) + 3\,J_{n+1}(x) - J_{n+3}(x).$$

11. Prove that $$J_2(x) = -\frac{1}{x}J_0'(x) + J_0''(x).$$

12. Prove that $$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

13. Prove that $$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).$$

14. Prove that $$J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right).$$

15. Prove that $$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right).$$

16. Prove that $$J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right).$$

17. Prove from (1), § 114, that
$$J_n(x+y) = \sum_{k=-\infty}^{k=\infty} J_k(x)J_{n-k}(x),$$
where n is an integer.

18. Prove by multiplying the expansions for $e^{\frac{z}{2}\left(t-\frac{1}{t}\right)}$ and $e^{-\frac{z}{2}\left(t-\frac{1}{t}\right)}$ that
$$[J_0(x)]^2 + 2[J_2(x)]^2 + 2[J_3(x)]^2 + \cdots = 1.$$

19. Verify formulas (11) and (12), § 117.

20. Show that the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta} + V = 0$$

is satisfied both by

$$V = \cos n\theta J_n(r) \quad \text{and by} \quad V = \sin n\theta J_n(r),$$

where n is an arbitrary constant.

21. Show that the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{r}\frac{\partial V}{\partial r} = 0$$

is satisfied both by

$$V = e^{\pm kz} \cos n\phi J_n(kr) \quad \text{and by} \quad V = e^{\pm kz} \sin n\phi J_n(kr),$$

where k and n are arbitrary constants.

22. Show that $\lim\limits_{n \to \infty} J_n(x) = 0$, and then show, by repeated use of (10), § 112, that

$$J_{n-1}(x) = \frac{2}{x}\left[nJ_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) - \cdots\right],$$

the series extending to infinity.

23. Show that

$$\frac{dJ_n(x)}{dx} = \frac{2}{x}\left[\frac{n}{2}J_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) - \cdots\right].$$

24. Show that $\qquad \lim\limits_{\epsilon \to 0} J_{-(n+\epsilon)}(x) = (-1)^n J_n(x),$

where n is an integer.

25. Show that $\qquad J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(x \cos \phi)\,d\phi.$

26. From Ex. 25 show that

$$\int_0^{\infty} e^{-ax} J_0(bx) = \frac{1}{\sqrt{a^2 + b^2}}.$$

CHAPTER XIII

PARTIAL DIFFERENTIAL EQUATIONS

118. Introduction. A partial differential equation is an equation which involves partial derivatives. In but comparatively few cases can the solutions of such an equation be written down explicitly. It is not the purpose of this text to discuss the theoretical questions involved in the study of partial differential equations, but merely to notice certain equations which are important in applied mathematics and to indicate certain methods for their solution. We shall accordingly leave untouched such questions as the proof of the existence of solutions, the convergence of series involved, and the validity of operations upon such series.

In general the solution of a partial differential equation involves arbitrary functions, just as the solution of an ordinary differential equation involves arbitrary constants. In a practical application the problem is usually to determine a particular function which will satisfy the differential equation and at the same time meet the other conditions of the practical problem.

119. Special forms of partial differential equations. Partial differential equations sometimes occur which can be readily solved by successive integration with respect to each of the variables, or which can be otherwise solved by elementary methods. No general discussion can very well be given for such equations, but the following examples will illustrate them:

Example 1. $\dfrac{\partial^2 z}{\partial x\, \partial y} = 0.$

By integration with respect to y we have

$$\frac{\partial z}{\partial x} = \phi_1(x),$$

where ϕ_1 is an arbitrary function. Integrating with respect to x, we have

$$z = \phi_2(x) + \phi_3(y),$$

where both ϕ_2 and ϕ_3 are arbitrary functions.

Example 2. $\dfrac{\partial^2 z}{\partial x^2} = -a^2 z.$

If x were the only independent variable, the solution of this equation would be

$$z = c_1 \sin ax + c_2 \cos ax.$$

This solution will also hold for the partial differential equation if we simply impose upon c_1 and c_2 the condition that they be independent of x but not necessarily independent of the other variables. That is, if z is a function of x and y, we have for the solution of the differential equation

$$z = \phi_1(y) \sin ax + \phi_2(y) \cos ax,$$

where $\phi_1(y)$ and $\phi_2(y)$ are arbitrary functions.

Example 3. $\dfrac{\partial^2 z}{\partial x^2} - a^2 \dfrac{\partial^2 z}{\partial y^2} = 0.$

If we place $x + ay = u$ and $x - ay = v$, the differential equation becomes

$$\frac{\partial^2 z}{\partial u\, \partial v} = 0,$$

the solution of which (Example 1) is

$$z = \phi_1(u) + \phi_2(v).$$

Hence the solution of the given equation is

$$z = \phi_1(x + ay) + \phi_2(x - ay).$$

When $a^2 = -1$, we have

$$z = \phi_1(x + iy) + \phi_2(x - iy)$$

as the solution of the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

120. The linear partial differential equation of the first order. This is the equation

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R, \qquad (1)$$

where P, Q, R are functions of x, y, z.

Let this equation be solved for $\dfrac{\partial z}{\partial x}$ so that

$$\frac{\partial z}{\partial x} = M + N\frac{\partial z}{\partial y}. \qquad (2)$$

By successive differentiation of (2) we may form all derivatives of the type

$$\frac{\partial^{r+s} z}{\partial x^r\, \partial y^s}$$

in terms of x, y, z and $\dfrac{\partial z}{\partial y}$, except that we cannot determine the partial derivatives $\dfrac{\partial^r z}{\partial y^r}$ with respect to y alone. We may therefore assume the values $x = x_0$, $y = y_0$, $z = z_0$, $\left(\dfrac{\partial z}{\partial y}\right)_0$, $\cdots$, $\left(\dfrac{\partial^r z}{\partial y^r}\right)_0$, $\cdots$ arbitrarily and compute the values of $\left(\dfrac{\partial^{r+s} z}{\partial x^r\, \partial y^s}\right)_0$ from these.

By using these results we have a Taylor series, as in § 38, which formally satisfies (1). It may be shown that this series will always converge in the neighborhood of (x_0, y_0, z_0) and therefore defines a solution of (1). The terms in y in this series define a function of y which is arbitrary, since the coefficients of y in the series expansion are arbitrary.

The discussion given above sketches the proof of the existence of the solution of (1) and makes clear why it is that the solution involves an arbitrary function. Actually to obtain the solution, the following method, based upon geometric considerations, is preferable:

Let
$$z = f(x, y) \tag{3}$$
be a solution of (1). The normal at any point to the surface defined by (3) has a direction $\dfrac{\partial z}{\partial x} : \dfrac{\partial z}{\partial y} : -1$ (§ 47), and (1) asserts that this direction is orthogonal to $P : Q : R$. That is, the values of P, Q, and R as determined at any point determine a direction on the surface (3). Hence if this direction is followed from point to point of (3), a curve is traced on (3). This curve is, however, a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{4}$$

Hence through every point of (3) there goes a curve of the family defined by (4). These curves are called *characteristics*, and we have shown that any solution of (1) is a locus of characteristics. Conversely, any surface which is a locus of characteristics will be a solution of (1), since its normal at each point is perpendicular to a curve of (4) and hence to $P : Q : R$.

The problem of solution is then reduced to that of grouping the curves (4) into surfaces. This may be done as follows:

Let
$$u(x, y, z) = c_1, \qquad v(x, y, z) = c_2 \tag{5}$$

be the solution of (4), where it is essential that the constants of integration should appear on the right of the equations. Then, if we form the equation

$$\phi(u, v) = 0, \tag{6}$$

where ϕ is an arbitrary function, we have the desired surfaces. For if in (5) we give c_1 and c_2 such values that

$$\phi(c_1, c_2) = 0, \tag{7}$$

the corresponding values of u and v satisfy (6) identically. That is, the curves (5) lie on (6).

We have, accordingly, the following method of procedure:

To solve equation (1), *first solve equations* (4) *for the characteristics and place the solution in the form* $u = c_1$, $v = c_2$. *Then*

$$\phi(u, v) = 0,$$

where ϕ *is an arbitrary function, is the solution of* (1).

For example, given

$$(ny - mz)\frac{\partial z}{\partial x} + (lz - nx)\frac{\partial z}{\partial y} = mx - ly. \tag{8}$$

We form

$$\frac{dx}{ny - mz} = \frac{dy}{lz - nx} = \frac{dz}{mx - ly}, \tag{9}$$

the solution of which is

$$x^2 + y^2 + z^2 = c_1, \qquad lx + my + nz = c_2. \tag{10}$$

Hence the solution of (7) is

$$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0. \tag{11}$$

From the discussion it follows that *a solution of* (1) *may be passed through any curve which is not a characteristic.* For if C is such a curve, equations (4) determine the direction of a characteristic at each point of C and therefore determine the whole characteristic through that point. The locus of all characteristics through C is a solution of (1).

The student will do well to notice the relation between the equations

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R, \tag{12}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \tag{13}$$

$$P\,dx + Q\,dy + R\,dz = 0, \tag{14}$$

with P, Q, R the same in all the equations. Equations (13) represent a family of curves, equation (12) represents surfaces made up of these curves, and equation (14) represents surfaces orthogonal to such curves provided the equation has a solution. Thus

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

represents straight lines through the origin,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

represents cones with vertex at the origin, and

$$x\,dx + y\,dy + z\,dz = 0$$

represents spheres with center at the origin.

When the family of curves (13) is given, the surfaces (12) always exist, but the surfaces (14) are not always possible.

121. The Fourier series. Before proceeding to the study of the differential equations of the next sections it is necessary for the student to have a working knowledge of the Fourier series. This is a series of the form

$$\begin{aligned}\frac{a_0}{2} &+ a_1 \cos x + a_2 \cos 2\,x + a_3 \cos 3\,x + \cdots \\ &+ b_1 \sin x + b_2 \sin 2\,x + b_3 \sin 3\,x + \cdots. \end{aligned} \tag{1}$$

We shall not concern ourselves with the theoretical questions involved in the careful study of this series, but shall give only a few practical rules for finding and using such series. It may be proved that it is possible to expand a function into such a series which will represent the function in an interval of length $2\,\pi$, provided the function is single-valued, finite, and continuous except for finite discontinuities and has not an infinite number of maxima or minima.

Let

$$\begin{aligned} f(x) = \frac{a_0}{2} &+ a_1 \cos x + a_2 \cos 2\,x + \cdots + a_k \cos kx + \cdots \\ &+ b_1 \sin x + b_2 \sin 2\,x + \cdots + b_k \sin kx + \cdots. \end{aligned} \tag{2}$$

Our problem is to determine the constants so that the series will represent the function in the interval $(-\pi, \pi)$.

To determine a_0 multiply (2) by dx and integrate from $-\pi$ to π, term by term. The result is

$$\int_{-\pi}^{\pi} f(x)dx = a_0\pi;$$

whence

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \tag{3}$$

c

since all the terms on the right-hand side of the equation, except the one involving a_0, vanish.

To obtain the other coefficients, we make use of the following elementary integrals, which may be obtained by direct integration:

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0, \\
&\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0 \quad \text{when} \quad m \neq n, \\
&\qquad\qquad\qquad\qquad\quad = \pi \quad \text{when} \quad m = n, \\
&\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0 \quad \text{when} \quad m \neq n, \\
&\qquad\qquad\qquad\qquad\quad = \pi \quad \text{when} \quad m = n.
\end{aligned} \tag{4}$$

Hence if we multiply (2) by $\cos kx\, dx$ and integrate between $-\pi$ and π, all terms on the right give 0 except the term involving a_k. Therefore

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx\, dx. \tag{5}$$

It is to be noted that (5) reduces to (3) when $k = 0$. Similarly, if we multiply (2) by $\sin kx\, dx$ and integrate between $-\pi$ and π, we have

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx\, dx. \tag{6}$$

For proof that the integration of the series is valid the student is referred to advanced treatises.

For example, let us expand e^x into a Fourier series in the interval $(-\pi, \pi)$. We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\, dx = \frac{e^\pi - e^{-\pi}}{\pi},$$

$$\begin{aligned}
a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos kx\, dx &= \left[\frac{e^x(k \sin kx + \cos kx)}{\pi(k^2+1)}\right]_{-\pi}^{\pi} \\
&= \frac{e^\pi - e^{-\pi}}{\pi(k^2+1)} \text{ when } k \text{ is even,} \\
&= -\frac{e^\pi - e^{-\pi}}{\pi(k^2+1)} \text{ when } k \text{ is odd,}
\end{aligned}$$

$$\begin{aligned}
b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin kx\, dx &= \left[\frac{e^x(\sin kx - k \cos kx)}{\pi(k^2+1)}\right]_{-\pi}^{\pi} \\
&= -\frac{k(e^\pi - e^{-\pi})}{\pi(k^2+1)} \text{ when } k \text{ is even,} \\
&= \frac{k(e^\pi - e^{-\pi})}{\pi(k^2+1)} \text{ when } k \text{ is odd.}
\end{aligned}$$

Hence

$$e^x = \frac{e^\pi - e^{-\pi}}{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos x + \frac{1}{5}\cos 2x - \frac{1}{10}\cos 3x + \cdots\right)$$
$$+ \frac{e^\pi - e^{-\pi}}{\pi}\left(\frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - \cdots\right). \qquad (7)$$

This series defines e^x only in the interval $(-\pi, \pi)$. Outside of this interval the values obtained from the series repeat themselves because of the periodic property of $\sin kx$ and $\cos kx$. Hence (as shown in Fig. 96) the series represents a periodic function which coincides with e^x only in the interval $(-\pi, \pi)$.

It is not necessary that $f(x)$ should be defined by a single equation in the entire interval $(-\pi, \pi)$. In case it is not so defined the integrals in (5) or (6) break up into two or more.

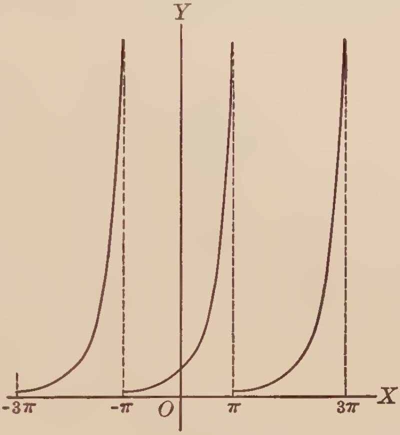

FIG. 96

For example, let

$$\begin{aligned} f(x) &= x + \pi \quad \text{when} \quad -\pi < x < 0, \\ f(x) &= \pi - x \quad \text{when} \quad 0 < x < \pi. \end{aligned} \qquad (8)$$

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)dx = \pi,$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\cos kx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos kx\,dx$$

$$= \frac{2}{\pi k^2}(1 - \cos kx) \qquad (k \neq 0)$$

$$= 0 \text{ when } k \text{ is even,}$$

$$= \frac{4}{k^2} \text{ when } k \text{ is odd,}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\sin kx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin kx\,dx = 0.$$

Hence the series for function (8) is

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3\,x}{3^2} + \frac{\cos 5\,x}{5^2} + \cdots\right), \tag{9}$$

and the graph of the series is shown in Fig. 97.

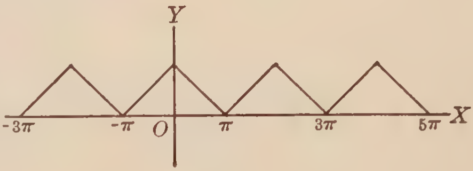

FIG. 97

It is also possible that the given function may have discontinuities in the interval $(-\pi, \pi)$. For example, let

$$\begin{aligned} f(x) &= -1 \quad \text{when} \quad -\pi < x < 0, \\ f(x) &= 1 \quad \text{when} \quad 0 < x < \pi. \end{aligned} \tag{10}$$

Then

$$a_k = \frac{1}{\pi}\int_{-\pi}^{0}(-\cos kx)dx + \frac{1}{\pi}\int_0^{\pi}\cos kx\,dx = 0,$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{0}(-\sin kx)dx + \frac{1}{\pi}\int_0^{\pi}\sin kx\,dx = \frac{2(1-\cos k\pi)}{\pi k} \\ &= 0 \text{ when } k \text{ is even,} \\ &= \frac{4}{\pi} \text{ when } k \text{ is odd.} \end{aligned}$$

Hence
$$f(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3\,x}{3} + \frac{\sin 5\,x}{5} + \cdots\right), \tag{11}$$

and the graph of the series is shown in Fig. 98.

In place of the interval $(-\pi, \pi)$ we may take any other interval of length $2\,\pi$, namely $(a, a + 2\,\pi)$, for formulas (4) remain valid if the limits of integration are taken as a and $a + 2\,\pi$. In that case the limits of integration in (5) and (6) are to be correspondingly changed.

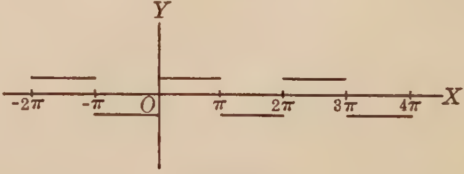

FIG. 98

Also, the interval in which the expansion takes place may be taken as $(-c, c)$ when c is any number. To show this we take

$$x = \frac{cy}{\pi}. \tag{12}$$

Then as y varies from $-\pi$ to π, x varies from $-c$ to c. We have

$$\begin{aligned} f(x) = f\left(\frac{cy}{\pi}\right) &= F(y) \\ &= \frac{a_0}{2} + a_1 \cos y + a_2 \cos 2y + \cdots \\ &\qquad + b_1 \sin y + b_2 \sin 2y + \cdots \\ &= \frac{a_0}{2} + a_1 \cos \frac{\pi x}{c} + a_2 \cos \frac{2\pi x}{c} + \cdots \\ &\qquad + b_1 \sin \frac{\pi x}{c} + b_2 \sin \frac{2\pi x}{c} + \cdots, \end{aligned} \tag{13}$$

where $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y) \cos ky\, dy = \frac{1}{c}\int_{-c}^{c} f(x) \cos \frac{k\pi x}{c}\, dx$

and $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y) \sin ky\, dy = \frac{1}{c}\int_{-c}^{c} f(x) \sin \frac{k\pi x}{c}\, dx.$ (14)

122. The Fourier series with sines or cosines only. If $f(x)$ is an even function, namely, if $f(-x) = f(x)$, then $f(x) \cos kx$ is also even and $f(x) \sin kx$ is odd. Hence

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos kx\, dx, \tag{1}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx\, dx = 0. \tag{2}$$

Hence an even function will be expanded into a series of cosine terms in the interval $-\pi < x < \pi$, and the coefficients may be computed by (1). An example occurs in series (9), § 121.

Also, if $f(x)$ is an odd function, namely if $f(-x) = -f(x)$, then $f(x) \sin kx$ is an even function and $f(x) \cos kx$ is an odd function. Hence

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx\, dx = 0, \tag{3}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx\, dx = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx\, dx. \tag{4}$$

Hence an odd function will be expanded into a series of sines in the interval $-\pi < x < \pi$, and the coefficients may be computed by (4).

From this it follows that any function $f(x)$ may be expanded in the interval from 0 to π into a series of cosines or a series of sines at pleasure. For we may, in the first place, define another function $F(x)$ which agrees with $f(x)$ for positive values of x and has for negative values of x the same values which $f(x)$ has for positive values of x. Then $F(x)$ is an even function and may be expanded into a series of cosines.

For example, consider $f(x) = e^x$ and let $F(x)$ be the function represented between $-\pi$ and π by Fig. 99, where the right-hand portion is the curve $y = e^x$ and the left-hand portion is symmetric to the right-hand portion with respect to OY. We may expand $F(x)$ into a series of cosines, using (1), by which

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^\pi e^x \cos kx\,dx \\ &= -\frac{2(e^\pi + 1)}{\pi(k^2+1)} \text{ when } k \text{ is odd,} \\ &= \frac{2(e^\pi - 1)}{\pi(k^2+1)} \text{ when } k \text{ is even.} \end{aligned} \tag{5}$$

Hence in the interval from 0 to π

$$\begin{aligned} e^x = {} & \frac{2(e^\pi - 1)}{\pi}\left(\frac{1}{2} + \frac{1}{5}\cos 2x + \frac{1}{17}\cos 4x + \cdots\right) \\ & - \frac{2(e^\pi + 1)}{\pi}\left(\frac{1}{2}\cos x + \frac{1}{10}\cos 3x + \frac{1}{26}\cos 5x + \cdots\right). \end{aligned} \tag{6}$$

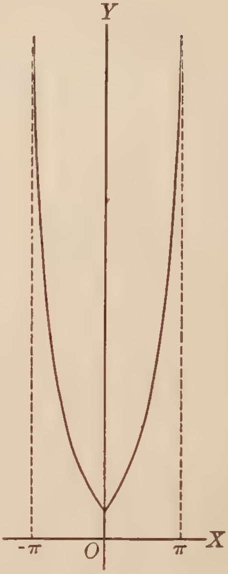

FIG. 99

In the second place, having given $f(x)$ between 0 and π, we may define $F(x)$ as equal to $f(x)$ between 0 and π and equal for $x = -a$ to the negative of the value of $f(x)$ for $x = a$. Then $F(x)$ is an odd function and may be expanded into a series of sines for which the coefficients are computed by (4).

For example, let $f(x) = e^x$, and let $F(x)$ be the function represented by Fig. 100, where the right-hand portion is the curve $y = e^x$ and the left-hand portion is symmetric to the left-hand portion with respect to O. We may expand $F(x)$ into a series of sines and, using (4), have

$$\begin{aligned} b_k &= \frac{2}{\pi}\int_0^\pi e^x \sin kx\,dx \\ &= \frac{2k(e^\pi + 1)}{\pi(k^2+1)} \text{ when } k \text{ is odd,} \\ &= -\frac{2k(e^\pi - 1)}{\pi(k^2+1)} \text{ when } k \text{ is even.} \end{aligned}$$

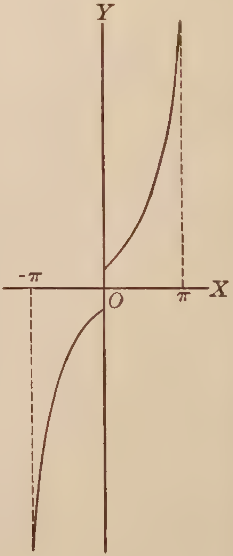

FIG. 100

Hence in the interval $(0, \pi)$ we have

$$e^x = \frac{2(e^\pi + 1)}{\pi}\left(\frac{1}{2}\sin x + \frac{3}{10}\sin 3x + \frac{5}{26}\sin 5x + \cdots\right)$$
$$-\frac{2(e^\pi - 1)}{\pi}\left(\frac{2}{5}\sin 2x + \frac{4}{17}\sin 4x + \frac{6}{37}\sin 6x + \cdots\right). \tag{7}$$

The student should compare series (6) and (7) with (7), § 121. Each series represents e^x in the interval $(0, \pi)$, but differs from each of the others in the function represented between $-\pi$ and 0.

123. Laplace's equation in two variables. An important equation in two variables x and y is the Laplace equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{1}$$

It has already been shown (§ 119) that the general solution of this equation is of the form

$$V = f_1(x + iy) + f_2(x - iy),$$

where f_1 and f_2 are arbitrary functions. But this is too general for practical use, as the difficulty of determining the functions to satisfy given conditions is too great.

The following method, however, has been found useful. The student should notice that this method is not used to find a general solution, but simply to find a particular solution which may satisfy given conditions. We place

$$V = XY, \tag{2}$$

where X is a function of x only and Y is a function of y only, and inquire if it is possible so to determine X and Y that (1) shall be satisfied.

By substitution (1) becomes, after a slight rearrangement,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}. \tag{3}$$

Now a change in x will not change the right-hand member of (3) and therefore will not change the left-hand member. Similarly, a change in y will not change the right-hand member of (3) and therefore will not change the left-hand member. Hence each of the two expressions in (3) is a constant which we shall denote by $-k^2$. Then (3) breaks up into two equations,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = k^2, \tag{4}$$

the solutions of which are, respectively,

$$X = c_1 \cos kx + c_2 \sin kx,$$
$$Y = c_3 e^{ky} + c_4 e^{-ky};$$

and hence

$$\begin{aligned} V &= (c_1 \cos kx + c_2 \sin kx)(c_3 e^{ky} + c_4 e^{-ky}) \\ &= e^{ky}(A \cos kx + B \sin kx) + e^{-ky}(M \cos kx + N \sin kx), \end{aligned} \tag{5}$$

where A, B, M, and N are arbitrary constants.

Now in (5) k may be given any value, fractional or integral. Let

$$V_0 = A_0 + M_0$$

be the solution (5) when $k = 0$,

$$V_1 = e^y(A_1 \cos x + B_1 \sin x) + e^{-y}(M_1 \cos x + N_1 \sin x)$$

be the solution (5) when $k = 1$,

$$V_2 = e^{2y}(A_2 \cos 2x + B_2 \sin 2x) + e^{-2y}(M_2 \cos 2x + N_2 \sin 2x)$$

be the solution (5) when $k = 2$, and so on. Then it is evident that

$$V = V_0 + V_1 + V_2 + \cdots + V_n \tag{6}$$

also satisfies (1). This is certainly true when n is finite. We shall assume that it is true when we take the limit as n increases indefinitely and (6) becomes an infinite series. This assumption of course needs proof; but in a practical problem we should, as a matter of fact, need only a few terms of (6), so that the matter is not of great practical importance. We have, then, as a solution of (1),

$$\begin{aligned} V = \sum_{n=0}^{n=\infty} [e^{ky}(A_k \cos kx + B_k \sin kx) \\ + e^{-ky}(M_k \cos kx + N_k \sin kx)]. \end{aligned} \tag{7}$$

It remains to determine the constants so as to satisfy given conditions. This will be illustrated in the next section.

If the coördinates x and y are replaced by polar coördinates (r, θ), equation (1) becomes

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r} = 0. \tag{8}$$

We may solve this by the method that we used in solving (1). Placing

$$V = R\Theta,$$

where R is a function of r only and Θ is a function of θ, and reasoning as before, we are led to the equations

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-k^2R=0,$$

and

$$\frac{d^2\Theta}{d\theta^2}=-k^2;$$

whence

$$R=c_1r^k+c_2r^{-k},$$

$$\Theta=c_3\cos k\theta+c_4\sin k\theta,$$

and, by a repetition of the former argument, we have a solution in the form

$$V=\sum_{k=0}^{k=\infty}[r^k(A_k\cos k\theta+B_k\sin k\theta)+r^{-k}(M_k\cos k\theta+N_k\sin k\theta)].$$

124. Application to flow of heat. Consider heat flowing in a medium. Let T be any region of the medium, let S be the boundary surface of T, where S is the geometric surface and not a physical one, and let θ be the temperature at any point of the medium. Then by Newton's law the amount of heat which flows in the time dt across an element dS is

$$-k\frac{d\theta}{dn}dS\,dt, \tag{1}$$

where $\frac{d\theta}{dn}$ is the rate of change of θ along the outwardly drawn normal to dS, and k is a constant, the conductivity of the medium. If $\frac{d\theta}{dn}$ is negative, the element (1) is positive and heat flows out; if $\frac{d\theta}{dn}$ is positive, heat flows in. The total amount of heat flowing across S is, then, the surface integral

$$-k\,dt\iint\limits_{(S)}\frac{d\theta}{dn}dS,$$

which, by § 35, is the same as

$$-k\,dt\iint\limits_{(S)}\left(\frac{\partial\theta}{\partial x}\cos\alpha+\frac{\partial\theta}{\partial y}\cos\beta+\frac{\partial\theta}{\partial z}\cos\gamma\right)dS,$$

which again, by § 79, is equal to the volume integral

$$-k\,dt\iiint\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right)dx\,dy\,dz \tag{2}$$

taken over T.

Now consider an element $dx\,dy\,dz$ of T. The amount of heat in this element at a given time t is

$$c\rho\theta\,dx\,dy\,dz,$$

where c is the specific heat and ρ is the density of the medium. After an interval dt the amount of heat in the element is

$$c\rho\left(\theta + \frac{\partial\theta}{\partial t}\,dt\right)dx\,dy\,dz,$$

and therefore the loss of heat in the element is

$$-\,c\rho\,\frac{\partial\theta}{\partial t}\,dx\,dy\,dz,$$

and the total loss of heat is

$$-\,c\rho\,dt\iiint\frac{\partial\theta}{\partial t}\,dx\,dy\,dz \tag{3}$$

taken over T.

Assuming that no heat is generated or destroyed in T, the expression (3) must equal (2), since the loss of heat inside is caused by the flow out across the surface. Hence, equating (2) and (3), canceling dt, and transposing, we have

$$\iiint\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}-h^2\,\frac{\partial\theta}{\partial t}\right)dx\,dy\,dz = 0, \tag{4}$$

where, for convenience, h^2 is put for the constant $\frac{c\rho}{k}$.

Now equation (4) is true for any region T whatever. Hence the integrand must vanish at each point, for if it did not we could take a region T in which it would be always positive or always negative (assuming continuity of the functions involved), and then the integral could not be zero.

Therefore we have as a fundamental equation for the flow of heat,

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2} = h^2\,\frac{\partial\theta}{\partial t}. \tag{5}$$

If the flow is steady, that is, independent of the time, then $\frac{\partial\theta}{\partial t} = 0$, and we have the Laplace equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2} = 0. \tag{6}$$

If, in addition, the flow takes place in planes parallel to XOY, θ is independent of z, $\frac{\partial \theta}{\partial z} = 0$, and we have

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0, \tag{7}$$

the equation of the last section.

We shall solve this equation for the special problem in which heat flows in a rectangular plate of breadth π and of infinite length, the end being kept at the temperature unity and the long edges at the temperature zero.

If we take the end of the plate as the axes of x and one of the long edges as the axis of y, we have to solve (7) subject to the conditions

$$\theta = 0 \quad \text{when} \quad x = 0, \tag{8}$$

$$\theta = 0 \quad \text{when} \quad x = \pi, \tag{9}$$

$$\theta = 1 \quad \text{when} \quad y = 0, \tag{10}$$

$$\theta = 0 \quad \text{when} \quad y = \infty, \tag{11}$$

the last condition (11) being introduced from the nature of the problem.

We pick up the solution (7), § 123, with V replaced by θ, and endeavor to determine the constants. Condition (11) shows at once that there can be no terms involving e^{ky}. Hence $A_k = 0$ and $B_k = 0$. Using condition (8), we have

$$0 = \sum_{k=0}^{k=\infty} M_k e^{-ky}$$

for all values of y. Hence $M_k = 0$ for all values of k. Our solution then reduces to

$$\theta = \sum_{k=0}^{k=\infty} N_k e^{-ky} \sin kx, \tag{12}$$

which satisfies (8), (9), and (11). It remains to satisfy condition (10), which gives

$$1 = \sum_{k=0}^{k=\infty} N_k \sin kx \tag{13}$$

to be valid for $0 < x < \pi$.

But this is a Fourier series for the expansion of 1, so that to obtain the coefficient we need to expand 1 in a sine series valid between 0 and π by the method of § 122. This gives

$$1 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3\,x + \frac{1}{5}\sin 5\,x + \cdots\right). \tag{14}$$

By comparison of (13) and (14) it appears that

$$N_k = \frac{4}{k\pi} \text{ when } k \text{ is odd}, \quad N_k = 0 \text{ when } k \text{ is even},$$

and hence the solution is shown to be

$$\theta = \frac{4}{\pi}\left(e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \cdots\right). \tag{15}$$

We have shown that (15) is a solution of the differential equation (7) which satisfies conditions (8) to (11). We have not shown that it is the only solution. That question must be postponed.

125. The Laplace equation in three variables. The Laplace equation in (x, y, z) coördinates is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{1}$$

We have already seen that this equation occurs in the flow of heat, and it occurs also in many other physical problems.

If (x, y, z) are replaced by cylindrical coördinates, (1) becomes

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0; \tag{2}$$

if (x, y, z) are replaced by polar coördinates, (1) becomes

$$r^2\frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \phi^2} + \cot\phi\frac{\partial V}{\partial \phi} + \csc^2\phi\frac{\partial^2 V}{\partial \theta^2} = 0. \tag{3}$$

We shall first consider equation (2) and attempt to solve it by placing

$$V = R\Theta Z, \tag{4}$$

where R is a function of r only, Θ is a function of θ only, and Z is a function of z only. By substitution and elementary reductions (2) becomes

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -\frac{1}{R}\frac{d^2R}{dr^2} - \frac{1}{rR}\frac{dR}{dr} - \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} = 0. \tag{5}$$

By hypothesis Z is a function of z only, and by (5) a change in z does not change $\frac{1}{Z}\frac{d^2Z}{dz^2}$, since it does not change the right-hand member of (5). Hence $\frac{1}{Z}\frac{d^2Z}{dz^2}$ reduces to a constant, and we write

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2; \tag{6}$$

whence

$$Z = c_1e^{kz} + c_2e^{-kz}. \tag{7}$$

Now from (5) and (6) we have

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+k^2r^2=-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}, \tag{8}$$

from which it follows that each member of (8) is a constant, which we take as m^2. We have, accordingly,

$$\frac{d^2\Theta}{d\theta^2}=-m^2\Theta; \tag{9}$$

whence

$$\Theta=c_3\cos m\theta+c_4\sin m\theta, \tag{10}$$

and

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(k^2r^2-m^2)R=0. \tag{11}$$

In (11) we place $kr=x$, and (11) becomes

$$x^2\frac{d^2R}{dx^2}+x\frac{d^2R}{dx}+(x^2-m^2)R=0, \tag{12}$$

which is a Bessel equation, so that

$$R=c_5J_m(x)+c_6J_{-m}(x)=c_5J_m(kr)+c_6J_{-m}(kr) \tag{13}$$

if m is fractional, or

$$R=c_5J_m(kr)+c_6K(kr) \tag{14}$$

if m is integral.

Any of these values of R, Θ, and Z substituted in (4) gives a solution V, and the sum of any number of such solutions is also a solution. In particular, let us assume k as a fixed constant and, letting m assume positive values, write the series

$$V=\sum_{m=0}^{m=\infty}[e^{kz}(A_m\cos m\theta+B_m\sin m\theta)$$
$$+e^{-kz}(C_m\cos m\theta+D_m\sin m\theta)]J_m(kr). \tag{15}$$

The sum of a finite number of terms of this series is a solution of (1), and we shall assume without proof that the limit of this sum as m increases is also a solution.

Consider next equation (3) under the assumption that the solution V is known to be symmetric about OZ. Then V is independent of θ, and hence $\frac{\partial V}{\partial\theta}=0$. We have the equation

$$\frac{\partial^2V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2V}{\partial\phi^2}+\frac{\cot\phi}{r^2}\frac{\partial V}{\partial\phi}=0. \tag{16}$$

Place
$$V = R\Phi, \tag{17}$$
where R is a function of r only and Φ a function of ϕ only. Proceeding as in the other case, we have eventually the two equations

$$r^2 \frac{d^2R}{dr^2} + 2\,r\frac{dR}{dr} - k^2R = 0, \tag{18}$$

$$\frac{d^2\Phi}{d\phi^2} + \cot\phi\,\frac{d\Phi}{d\phi} + k^2\Phi = 0, \tag{19}$$

where k is any constant.

Equation (18) may be solved by the method of § 107, with the result
$$R = c_1 r^{-\frac{1}{2}+\sqrt{k^2+\frac{1}{4}}} + c_2 r^{-\frac{1}{2}-\sqrt{k^2+\frac{1}{4}}}. \tag{20}$$

It will then be convenient to replace k by m, where
$$m = -\tfrac{1}{2} + \sqrt{k^2+\tfrac{1}{4}}.$$

Then (20) becomes
$$R = c_1 r^m + \frac{c_2}{r^{m+1}}, \tag{21}$$

and (19) becomes

$$\frac{d^2\Phi}{d\phi^2} + \cot\phi\,\frac{d\Phi}{d\phi} + m(m+1)\Phi = 0. \tag{22}$$

In this equation place $t = \cos\phi$, and it becomes

$$(1-t^2)\frac{d^2\Phi}{dt^2} - 2\,t\frac{d\Phi}{dt} + m(m+1)\Phi = 0, \tag{23}$$

a Legendre equation. If m is taken as a positive integer, (23) is solved by the Legendre polynomial

$$\Phi = P_m(t) = P_m(\cos\phi). \tag{24}$$

By combining the solutions thus obtained we have for (16) a solution
$$V = \sum_{m=0}^{m=\infty}\left(A_m r^m + \frac{B_m}{r^{m+1}}\right)P_m(\cos\phi). \tag{25}$$

An application of this result will be shown in the next section.

126. Application to potential. Let a particle of matter of mass m be at the point $(a,\ b,\ c)$. Then the gravitational potential at $(x,\ y,\ z)$ due to the mass m is

$$\frac{m}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}.$$

Consider now any solid T of density ρ. Then $\rho\, da\, db\, dc = dm$, the mass of an element of volume $da\, db\, dc$. The potential V due to the mass is the sum of the potentials of the individual particles and is the volume integral

$$V = \iiint \frac{\rho\, da\, db\, dc}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}} \tag{1}$$

taken throughout T. If (x, y, z) is not inside T, the integral is continuous in T and may be differentiated under the integral sign.

It is not difficult in this way to show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{2}$$

so that the potential function satisfies Laplace's equation.

Let us apply this to the problem of finding the potential at any point due to a circular ring of small cross section and of radius a, lying in the plane XOY with center at O. Since the solution is obviously symmetric about OZ, we will replace (x, y, z) by polar coördinates and use the solution

$$V = \sum_{m=0}^{m=\infty} \left(A_m r^m + \frac{B_m}{r^{m+1}} \right) P_m(\cos \phi), \tag{3}$$

found in the previous section. The problem is to determine the coefficients. This may be done by noticing that any point Q on OZ is at the same distance $\sqrt{a^2+r^2}$ from all points of the ring where $OQ = r$, the dimensions of the ring being negligible. Hence the potential at Q is

$$\frac{M}{\sqrt{a^2+r^2}}, \tag{4}$$

where M is the total mass of the ring.

Now (4) may be expanded by the binomial theorem into the convergent series

$$\frac{M}{a}\left(1 - \frac{1}{2}\frac{r^2}{a^2} + \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{a^4} - \cdots\right) \quad \text{when} \quad r < a, \tag{5}$$

and into the convergent series

$$\frac{M}{a}\left(\frac{a}{r} - \frac{1}{2}\frac{a^3}{r^3} + \frac{1\cdot 3}{2\cdot 4}\frac{a^5}{r^5} - \cdots\right) \quad \text{when} \quad r > a. \tag{6}$$

The general solution (3) must reduce to (5) or (6) for a point on OZ. For such a point Q, we have $\phi = 0$, $P_m(\cos\phi) = P_m(1) = 1$, and (3) becomes

$$V = \sum_{m=0}^{m=\infty}\left(A_m r^m + \frac{B_m}{r^{m+1}}\right),$$

which must be either (5) or (6). Hence if $r < a$, $B_m = 0$, and A_m are the coefficients in (5); if $r > a$, $A_m = 0$, and B_m are the coefficients of (6). Hence we have the solution

$$V = \frac{M}{a}\left[P_0(\cos\phi) - \frac{1}{2}\frac{r^2}{a^2}P_2(\cos\phi) + \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{a^4}P_4(\cos\phi) - \cdots\right]$$

when $r < a$, and the solution

$$V = \frac{M}{a}\left[\frac{a}{r}P_0(\cos\phi) - \frac{1}{2}\frac{a^3}{r^3}P_2(\cos\phi) + \frac{1\cdot 3}{2\cdot 4}\frac{a^5}{r^5}P_4(\cos\phi) - \cdots\right]$$

when $r > a$.

127. Harmonic functions. A function $V(x, y)$ which, together with its derivatives of the first and second order, is continuous, except for definite points called singularities, and which satisfies the Laplace equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1}$$

is a harmonic function in the plane. We have shown in § 73 that if C is the boundary of a region in which V has no singularities,

$$\int_{(C)}\frac{dV}{dn}\,ds = 0. \tag{2}$$

Similarly, a function $V(x, y, z)$ which satisfies the Laplace equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{3}$$

together with the same conditions as to continuity, is a harmonic function in space. We have shown in § 79 that if S is the boundary of a region in which V has no singularities, then

$$\iint_{(S)}\frac{dV}{dn}\,dS = 0. \tag{4}$$

From equations (2) and (4) certain important properties of harmonic functions may be deduced which are similar in the plane and in space. We shall consider the case for the plane, leaving for the student the similar case in space.

We shall apply (2) to a circle of radius r and center A. Here $\dfrac{dV}{dn} = \dfrac{\partial V}{\partial r}$, $ds = r\,d\theta$, and (2) becomes

$$\int_0^{2\pi} \frac{\partial V}{\partial r}\,d\theta = 0, \tag{5}$$

since r, being constant in the integration, may be divided out. Let (5) be multiplied by dr and the result be integrated between $r = 0$ and $r = a$. Then we have

$$\int_0^a dr \int_0^{2\pi} \frac{\partial V}{\partial r}\,d\theta = \int_0^{2\pi} d\theta \int_0^a \frac{\partial V}{\partial r}\,dr = 0, \tag{6}$$

which gives

$$\int_0^{2\pi} (V_a - V_0)d\theta = 0, \tag{7}$$

where V_a is the value of V when $r = a$ and in general depends upon θ, and V_0 is the value of V_0 at the point A and is independent of V. Hence (7) may be written

$$2\,\pi V_0 = \int_0^{2\pi} V_a\,d\theta,$$

or

$$V_0 = \overline{V}_a, \tag{8}$$

where

$$\overline{V}_a = \frac{1}{2\,\pi}\int_0^{2\pi} V_a\,d\theta$$

and is the average value of V on the circumference of radius a. Hence we have our first theorem:

I. The average value of a harmonic function on the circumference of a circle in which it has no singularities is equal to its value at the center of the circle.

This theorem is made valid in space by replacing the circle by a sphere. The proof is left to the student.

From this we may deduce the following theorems, which are so stated as to be valid in either two or three dimensions:

II. A harmonic function without singularities in a given region cannot have a maximum value or a minimum value in the region.

To prove this let us suppose for a moment that V has a maximum value V_0 at a point A. Then we may draw a small circle around A such that $V_0 > V_a$ for all points on this circle. Then

$$\overline{V}_a = \frac{1}{2\,\pi}\int_0^{2\pi} V_a\,d\theta < \frac{1}{2\,\pi}\int_0^{2\pi} V_0\,d\theta, \qquad \text{or} \qquad \overline{V}_a < V_0,$$

which contradicts theorem I. Hence V cannot have a maximum value in the region, and in a similar manner it cannot have a minimum value there.

It follows that the maximum and minimum values of V must occur on the boundary of the region. Hence if the value of V is constant on the boundary, its maximum and minimum values coincide, and the function is constant. This gives us the following theorem:

III. A harmonic function with no singularities within a region and with constant values on the boundary of the region is constant throughout.

Suppose now we have two harmonic functions V_1 and V_2 which have the same values on the boundary of a closed region. Then $V_1 - V_2$ is a harmonic function which is zero on the boundary and hence, by theorem III, is zero throughout. Hence we have the following theorem:

IV. Two harmonic functions which have identical values upon a closed contour and have no singularities within the contour are identical throughout the region bounded by the contour.

One practical result of this theorem is that if a solution of Laplace's equation has been found by the empirical methods of the previous sections so as to take assigned values on a closed boundary, no other solution is possible.

Let us now in (2), § 79 (written for two dimensions), place $G = F = V$, where V is a harmonic function. We get

$$\int V \frac{dV}{dn}\, ds = \iint \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right] dx\, dy. \tag{9}$$

If $\dfrac{dV}{dn} = 0$ along a contour C, and V is a real function of x and y, we must have $\dfrac{\partial V}{\partial x} = 0$, $\dfrac{\partial V}{\partial y} = 0$; that is, V is a constant. Hence we have the theorem:

V. If the normal derivative of a harmonic function is zero along a closed contour within which the function has no singularities, the function is constant.

From this follows the theorem:

VI. If two harmonic functions have the same normal derivative along a closed contour within which they have no singularities, they differ at most by an additive constant.

Applied to the flow of heat, theorems IV and VI are almost immediately evident. One says that the temperature within a closed region is fully determined by the temperature on the boundary, and the other says that except for an additive constant the temperature inside a region is determined by the rate of flow across the boundary.

EXERCISES

Solve the following equations:

1. $\frac{\partial^2 z}{\partial x^2} = a^2 z.$

2. $x\frac{\partial^2 z}{\partial x^2} = a\frac{\partial z}{\partial x}.$

3. $\frac{\partial^2 z}{\partial x^2} - a\frac{\partial z}{\partial x} = 0.$

4. $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial z}{\partial x} + 2z = 0.$

5. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial z}{\partial x} + z = 0.$

6. $\frac{\partial^2 z}{\partial x^2} = x + y.$

7. $\frac{\partial^2 z}{\partial x^2} = x^2 y^2.$

8. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1.$

9. $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0.$

10. $2x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = z.$

11. $(z + x)\frac{\partial z}{\partial y} - (z + y)\frac{\partial z}{\partial x} = y - x.$

12. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$

13. $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 1.$

14. $xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = xy.$

15. Show that $a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1$ is an equation satisfied by all cylinders the elements of which are parallel to a fixed direction.

16. Show that $(x - a)\frac{\partial z}{\partial x} + (y - b)\frac{\partial z}{\partial y} = z - c$ is the differential equation of all cones the vertices of which are at a fixed point.

17. Show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$ is the differential equation of surfaces generated by lines parallel to a fixed plane and intersecting a fixed normal to that plane.

18. Show that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$ is the differential equation of surfaces of revolution with OZ as axis.

19. Show that in cylindrical coördinates $\theta \dfrac{\partial z}{\partial \theta} = z$ is a differential equation of surfaces the characteristics of which are helices.

20. Show that $2\,xy \dfrac{\partial z}{\partial x} + (y^2 - x^2) \dfrac{\partial z}{\partial y} = 0$ is the differential equation of certain surfaces the characteristics of which are circles in parallel planes.

Expand the following functions into Fourier series in the interval $(-\pi, \pi)$:

21. $f(x) = x$. **22.** $f(x) = x^2$. **23.** $f(x) = x^3$.

24. $f(x) = 0$ when $-\pi < x < 0$, $f(x) = \pi$ when $0 < x < \pi$.

25. $f(x) = -x$ when $-\pi < x < 0$, $f(x) = 0$ when $0 < x < \pi$.

26. $f(x) = -\pi$ when $-\pi < x < 0$, $f(x) = x$ when $0 < x < \pi$.

27. $f(x) = 0$ when $-\pi < x < 0$, $f(x) = x^2$ when $0 < x < \pi$.

Expand the following functions into Fourier series in the interval $(0, 2\pi)$:

28. $f(x) = x$. **29.** $f(x) = x^2$. **30.** $f(x) = x^3$.

31. $f(x) = 1$ when $0 < x < \pi$, $f(x) = 0$ when $\pi < x < 2\pi$.

32. $f(x) = x$ when $0 < x < \pi$, $f(x) = 2\pi - x$ when $\pi < x < 2\pi$.

33. Expand $f(x) = 1$ into a sine series in the interval $(0, \pi)$.

34. Expand $f(x) = x$ into a cosine series in the interval $(0, \pi)$.

35. Expand $f(x) = x^2$ into a sine series in the interval $(0, \pi)$.

36. Expand $f(x) = \sin x$ into a cosine series in the interval $(0, \pi)$.

37. Expand $f(x) = \cos x$ into a sine series in the interval $(0, \pi)$.

38. Solve equation (5), § 124, under the assumption that the flow is steady and takes place radially outward from the center of a sphere.

39. Solve equation (5), § 124, under the assumption that the flow of heat is steady and takes place radially outward from the axis of a cylinder.

40. Find a particular solution of the equation $\dfrac{\partial y}{\partial t} = a^2 \dfrac{\partial^2 y}{\partial x^2}$.

41. Find a particular solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = h^2 \frac{\partial V}{\partial t}.$$

42. Find the temperature for a steady flow of heat in a semicircular plate, the circumference being kept at a temperature 1 and the diameter at a temperature 0.

43. If a slab of thickness c is originally at a temperature unity throughout and both sides are then kept at a temperature zero, find the temperature at any point of the slab, the slab being so large that only the flow normal to its faces need be considered.

44. Find the temperature for a steady flow of heat in a circular plate if half the circumference is kept at a temperature 0 and the other half at a temperature 1.

45. Solve the equation

$$\frac{\partial \theta}{\partial t} = h^2 \frac{\partial^2 \theta}{\partial x^2} - b^2 \theta,$$

which is that for the flow of heat in a long rod with radiating surfaces, assuming that when $t = 0$, $\theta = f(x)$.

46. A vibrating string may be shown to satisfy the equation

$$a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2},$$

where x and y are the coördinates of a point on the string and t is time. Solve the equation if at a time $t = 0$ the string has a position $y = f(x)$. Take the length of the string as l and assume that the ends are fixed.

47. An oscillating chain hanging vertically may be shown to satisfy the equation

$$\frac{\partial^2 y}{\partial t^2} = g(l - x) \frac{\partial^2 y}{\partial x^2} - g \frac{\partial y}{\partial x},$$

where (x, y) are the coördinates of a point on the chain, l is its length, and t is time. Find a possible solution.

48. A cross section of the surface of a wave in a bay satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{g}{b} \frac{\partial}{\partial x}\left(hb \frac{\partial y}{dx}\right),$$

where b is the breadth of the bay, h is its depth, x is a horizontal axis running out to sea, and y is a vertical axis. Find a solution, assuming $h =$ constant and $b = kx$.

49. Solve Ex. 48, assuming $b =$ constant and $h = kx$.

50. Find the potential due to a homogeneous circular disk of radius a and mass M, first finding by ordinary integration the potential due to the ring at any point of a line perpendicular to the disk at its center.

51. Show that equation (3), § 125, has a solution of the form

$$V = \sum_{n=0}^{n=\infty} \sum_{m=0}^{m=\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}}\right)(a_m \cos m\theta + b_m \sin m\theta) P_n{}^m(\cos \phi),$$

where $P_n{}^m$ is an associated Legendre function.

52. Show that the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + V = 0$$

has a solution in polar coördinates of the form

$$V = \sum_{n=0}^{n=\infty} (a_n \cos n\theta + b_n \sin n\theta) J_n(r).$$

53. Show that the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + V = 0$$

has a solution in polar coördinates of the form

$$V = \sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(r) (A_m \cos m\theta + B_m \sin m\theta) P_n{}^m(\cos \phi).$$

54. State and prove for space the theorems of § 127.

CHAPTER XIV

CALCULUS OF VARIATIONS

128. The simplest case. Consider the integral

$$\int_a^b f\left(x,\, y,\, \frac{dy}{dx}\right)dx \tag{1}$$

taken along a curve

$$y = \phi(x) \tag{2}$$

connecting two points A and B in the plane (Fig. 101). The value of the integral depends in general upon the curve, and we wish to determine the effect of varying the curve and, in particular, to find the curve which makes the value of the integral a maximum or a minimum. For that purpose we will call the curve (2) the original curve C, and the curve

$$Y = \phi(x) + \eta(x) \tag{3}$$

the varied curve C'. Then

$$Y - y = \eta(x),$$

represented by QP, is the variation of y and will be denoted by δy, so that (3) becomes

$$Y = y + \delta y. \tag{4}$$

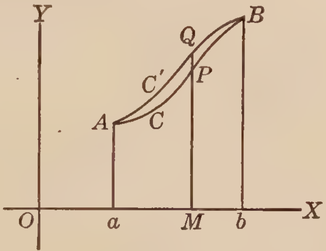

FIG. 101

Denote $\dfrac{dy}{dx}$ by y' and $\dfrac{dY}{dx}$ by Y'. Then, by (3),

$$Y' = y' + \eta'(x) = y' + \frac{d}{dx}(\delta y). \tag{5}$$

We shall call $\eta'(x)$ the variation of y' and denote it by $\delta y'$. It follows from (5) that

$$\delta y' = \frac{d}{dx}(\delta y), \tag{6}$$

or, otherwise written,

$$\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y), \tag{7}$$

a formula which shows the allowable interchange of d and δ.

We shall assume that the quantities δy and $\delta y'$ are both small; that is, that the height and the slope of the curve C' at any point differ very little from the height and the slope of the curve C at the corresponding point.

Consider now the function $f(x, y, y')$ for any point of the curve C. For the corresponding point of C' the function becomes

$$f(x, y + \delta y, y' + \delta y') = f(x, y, y') + \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial y'}\,\delta y' + \cdots,$$

the expansion being made by Taylor's series. We shall call the sum of the terms of first order in δy and $\delta y'$ the first variation of f and denote it by δf, so that we have

$$\delta f = \frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial y'}\,\delta y'. \tag{8}$$

Let I be the value of the integral (1) along the curve C. Its value along the varied curve C' is found by replacing $f(x, y, y')$ by $f(x, y + \delta y, y' + \delta y')$. We shall call that part of the change in the integral which contains only terms of the first order in δy and $\delta y'$ the first variation of the integral and denote it by δI.

Then, by (8),
$$\delta I = \int_a^b \left(\frac{\partial f}{\partial y}\,\delta y + \frac{\partial f}{\partial y'}\,\delta y'\right) dx. \tag{9}$$

Consider the second part of the integral in (9). By (6) we have

$$\int_a^b \frac{\partial f}{\partial y'}\,\delta y'\,dx = \int_a^b \frac{\partial f}{\partial y'}\frac{d}{dx}\,(\delta y)dx,$$

and by integration by parts

$$\int_a^b \frac{\partial f}{\partial y'}\,\delta y'\,dx = \left[\frac{\partial f}{\partial y'}\,\delta y\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y\,dx.$$

But since the points A and B are not changed, $\delta y = 0$ when $x = a$ and $x = b$, so that

$$\int_a^b \frac{\partial f}{\partial y'}\,\delta y'\,dx = -\int_a^b \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y\,dx. \tag{10}$$

Substituting this value in (9), we have

$$\delta I = \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y\,dx. \tag{11}$$

This, then, is the change in I as far as the first order of infinitesimals δy and $\delta y'$ is concerned.

Now if δy and $\delta y'$ are sufficiently small, it is obvious that the sign of the exact change in I will be determined by the sign of δI, since the terms δy^2, $\delta y'^2$, $\delta y\,\delta y'$, etc. are of higher order than

δy (there is no difficulty in making such variations that $\delta y'$ is of the same order as δy).

Hence if δI is positive or negative for a certain variation δy, its sign would be reversed for a variation $-\delta y$. In that case I would be increased by one variation and decreased by another. From this it follows that if I is to have a maximum or a minimum value along the curve C, it is necessary that

$$\delta I = 0 \tag{12}$$

for all possible small variations δy.

From this it follows that the coefficient of δy in (11) should be zero at all points of the curve. For if this expression were not zero we might so vary the curve C that δy should have the same sign as its coefficient, and then δI would be positive, in contradiction to (12). Hence we have established the theorem:

If I has a maximum or a minimum value along a curve C, that curve must be a solution of the differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{13}$$

In formula (13) the partial derivatives indicate merely formal operations under the assumption that x, y, and y' are independent variables. The operation $\frac{d}{dx}$, however, takes into account that y and y' are functions of x. Hence (13) may be transformed into

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x\,\partial y'} - \frac{\partial^2 f}{\partial y\,\partial y'}\,y' - \frac{\partial^2 f}{\partial y'^2}\,y'' = 0. \tag{14}$$

This is an equation of the second order to determine y. The solution contains two arbitrary constants, and it is necessary, if possible, so to determine these that the curve shall pass through the given points A and B.

When this has been done we can assert that if the integral has a maximum or a minimum value, it must be obtained along this curve. The question as to whether the maximum or the minimum value actually exists, and if so which one it is, is still unanswered. In mathematical language, we have determined a *necessary* but not a *sufficient* condition. The determination of the sufficiency of the condition is a matter of too great complexity for this text. In practical problems the question can often be decided from the nature of the problem.

As an example of the method, consider the problem of finding the curve between given points A and B which, by revolution about OX, generates the surface of least area. By elementary calculus the area in question is given by

$$S = 2\pi \int_a^b y\,ds = 2\pi \int_a^b y\sqrt{1+y'^2}\,dx.$$

Here $f = y\sqrt{1+y'^2}$, $\quad \dfrac{\partial f}{\partial y} = \sqrt{1+y'^2}$, $\quad \dfrac{\partial f}{\partial y'} = \dfrac{yy'}{\sqrt{1+y'^2}}$.

Therefore equation (13) is

$$\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0,$$

which reduces to $\qquad 1 + y'^2 - yy'y'' = 0 \qquad (15)$

as the form which (14) takes.

To integrate (15), place $y' = p$ and $y'' = \dfrac{dp}{dy}\cdot$ The equation is then

$$\frac{p\,dp}{1+p^2} = \frac{dy}{y};$$

whence, finally, $\qquad y = c_1 \cosh \dfrac{x - c_2}{c_1}, \qquad (16)$

the equation of the catenary. The constants c_1 and c_2 must now be determined so that the curve (16) passes through A and B. The question as to whether this is possible and whether, if so, there is a maximum or a minimum will not be considered here. It is physically evident that in many cases the solution exists.

129. Solution by differentials. We may write

$$f(x, y, y')dx = \phi(x, y, dx, dy) \qquad (1)$$

and consider the integral

$$\int_{(A)}^{(B)} \phi(x, y, dx, dy) \qquad (2)$$

along a curve between fixed points A and B.

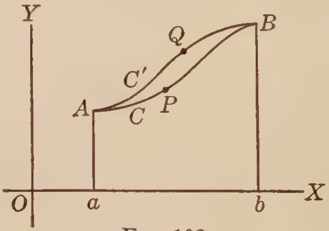

FIG. 102

In the varied curve the point P may be considered displaced to Q so that x and y take each a variation (Fig. 102). Hence if (x, y) are the coördinates of P, and (X, Y) those of Q,

$$X = x + \delta x, \qquad Y = y + \delta y; \qquad (3)$$

whence $\qquad dX = dx + d(\delta x), \qquad dY = dy + d(\delta y),$

and therefore $\qquad \delta(dx) = d(\delta x), \qquad \delta(dy) = d(\delta y). \qquad (4)$

Then, as in § 128,

$$\delta\phi = \frac{\partial\phi}{\partial x}\,\delta x + \frac{\partial\phi}{\partial y}\,\delta y + \frac{\partial\phi}{\partial(dx)}\,\delta(dx) + \frac{\partial\phi}{\partial(dy)}\,\delta(dy), \tag{5}$$

and

$$\delta I = \int_{(A)}^{(B)} \delta\phi, \tag{6}$$

where in (5) the partial derivatives are to be taken on the hypothesis that x, y, dx, dy are independent variables.

In (5) replace $\delta(dx)$ and $\delta(dy)$ by their values as given by (4), and integrate by parts the corresponding terms of (6). There results

$$\begin{aligned}\delta I = &\left[\frac{\partial\phi}{\partial(dx)}\,\delta x + \frac{\partial\phi}{\partial(dy)}\,\delta y\right]_{(A)}^{(B)} \\ &+ \int_{(A)}^{(B)} \left\{\left[\frac{\partial\phi}{\partial x} - d\left(\frac{\partial\phi}{\partial(dx)}\right)\right]\delta x + \left[\frac{\partial\phi}{\partial y} - d\left(\frac{\partial\phi}{\partial(dy)}\right)\right]\delta y\right\}. \end{aligned} \tag{7}$$

Since by hypothesis δx and δy are zero at A and B the quantity in the first pair of brackets vanishes. Arguing then as in § 128, we can easily see that for maximum or minimum values of I we must have each of the following relations satisfied:

$$\frac{\partial\phi}{\partial x} - d\left(\frac{\partial\phi}{\partial(dx)}\right) = 0, \tag{8}$$

$$\frac{\partial\phi}{\partial y} - d\left(\frac{\partial\phi}{\partial(dy)}\right) = 0. \tag{9}$$

As a matter of fact, we have here one relation and not two, for it is possible to show that each of the equations (8) and (9) is equivalent to (13) of § 128. This may be done as follows: From (1)

$$\frac{\partial\phi}{\partial x} = \frac{\partial f}{\partial x}\,dx,$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial f}{\partial y}\,dx,$$

$$\frac{\partial\phi}{\partial(dx)} = f + \frac{\partial f}{\partial y'}\,\frac{\partial y'}{\partial(dx)}\,dx = f - \frac{\partial f}{\partial y'}\,\frac{dy}{dx},$$

$$\frac{\partial\phi}{\partial(dy)} = \frac{\partial f}{\partial y'}\,\frac{\partial y'}{\partial(dy)}\,dx = \frac{\partial f}{\partial y'}.$$

Substituting these values in (8), we have

$$\frac{\partial f}{\partial x}dx - d\left(f - \frac{\partial f}{\partial y'}y'\right) = 0,$$

$$\frac{\partial f}{\partial x}dx - \left[\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial y'}dy' - d\left(\frac{\partial f}{\partial y'}\right)y' - \frac{\partial f}{\partial y'}dy'\right] = 0,$$

which is
$$\frac{\partial f}{\partial y}dy - d\left(\frac{\partial f}{\partial y'}\right)\frac{dy}{dx} = 0.$$

Dividing by dy, we have (13) of § 128.

Again, substituting in (9), we have

$$\frac{\partial f}{\partial y}dx - d\left(\frac{\partial f}{\partial y'}\right) = 0,$$

and dividing by dx, we have again (13), § 128.

The two equations (8) and (9), while not giving a new condition, give us two new forms, one of which may be more convenient than the other or than (13), § 128.

For example, consider again the integral used in § 128 written in the form

$$\int y\sqrt{dx^2 + dy^2}.$$

Equation (8) is now

$$0 - d\left(\frac{y\,dx}{\sqrt{dx^2 + dy^2}}\right) = 0,$$

and (9) is
$$\sqrt{dx^2 + dy^2} - d\left(\frac{y\,dy}{\sqrt{dx^2 + dy^2}}\right) = 0.$$

The first of these is the simpler and gives at once

$$\frac{y\,dx}{\sqrt{dx^2 + dy^2}} = c_1;$$

whence
$$\frac{c_1\,dy}{\sqrt{y^2 - c_1{}^2}} = dx,$$

and
$$y = c_1 \cosh\frac{x - c_2}{c_1},$$

as before.

An application to the determination of geodesics on a surface is of interest. Let the equation of a surface be

$$x = f(u, v), \qquad y = f(u, v), \qquad z = f(u, v), \tag{10}$$

where (u, v) are curvilinear coördinates on the surface. The length of any space curve is

$$s = \int \sqrt{dx^2 + dy^2 + dz^2};$$

and if the curve lies on the surface (10), dx, dy, dz may be replaced by their values obtained from (10), so that

$$s = \int \sqrt{E\,du^2 + 2\,F\,du\,dv + G\,dv^2}, \tag{11}$$

where E, F, G are as in § 53.

The lines which make the first variation of this integral vanish are by definition the *geodesics* on the surface. They will be the lines of shortest length between two points not too far remote.

130. Variable limits. We will now suppose that the points A and B, which were held fixed in §§ 128 and 129, are allowed to vary along two fixed curves L_1 and L_2 (Fig. 103). That is, we ask what curve C, the extremities of which lie anywhere on L_1 and L_2, will make the integral

$$\int \phi(x, y, dx, dy)$$

a maximum or a minimum.

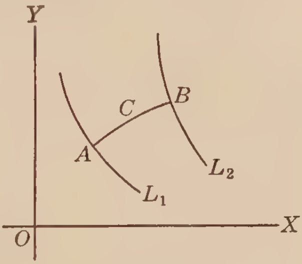

FIG. 103

In the first place, it is evident that that curve must satisfy equations (8) and (9) of § 129, for among all the curves which may be drawn between L_1 and L_2 are the curves which leave A and B fixed.

The variation (7) of § 129 therefore becomes

$$\delta I = \left[\frac{\partial \phi}{\partial(dx)}\,\delta x + \frac{\partial \phi}{\partial(dy)}\,\delta y\right]_{(A)}^{(B)},$$

and since this must vanish we have the condition

$$\frac{\partial \phi}{\partial(dx)}\,\delta x + \frac{\partial \phi}{\partial(dy)}\,\delta y = 0 \tag{1}$$

to be satisfied at each end of the curve C.

In (1) dx and dy are determined by the direction of C, and δx and δy are determined by the direction of L_1 or L_2, as the case may be. Hence (1) gives a relation between the direction of C and that of L_1 and L_2 at the points where C intersects L_1 and L_2.

As an example, consider the problem of finding the shortest distance between any two curves L_1 and L_2. We are to minimize the integral

$$\int \sqrt{dx^2 + dy^2}. \tag{2}$$

Equation (8) of § 129 becomes

$$-d \frac{dx}{\sqrt{dx^2 + dy^2}} = 0,$$

the solution of which is $\quad y = c_1 x + c_2. \qquad (3)$

Equation (1) reduces to

$$dx\,\delta x + dy\,\delta y = 0,$$

which says that the straight line (3) must cut the curves L_1 and L_2 at right angles.

Hence if the shortest distance exists, that distance will be the length of the straight line cutting both curves at right angles. Nothing in our work, however, proves that any straight line which satisfies these conditions is a solution of the problem. A simple example will show this. Consider two circles tangent internally (Fig. 104). The line ABC is the only line perpendicular to both circles, but neither the segment AB nor the segment BC is the shortest distance between the two circles. We may, of course, consider the zero segment CC as a piece of the straight line ABC.

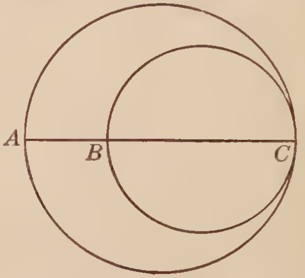

FIG. 104

131. Constrained variation. Let it be required to make the integral

$$I = \int \phi(x, y, dx, dy) \tag{1}$$

a maximum or a minimum while keeping the integral

$$J = \int \psi(x, y, dx, dy) \tag{2}$$

equal to a constant a. According to our previous discussion we must have

$$\delta I = 0, \tag{3}$$

but are now to admit only variations δx, δy, $\delta(dx)$, and $\delta(dy)$ for which

$$\delta J = 0, \tag{4}$$

since J is not to change its value. But if (3) and (4) are satisfied, it is necessary (though not sufficient) that

$$\delta I + \lambda \delta J = 0, \tag{5}$$

where λ is any constant multiplier; that is, the integral

$$I + \lambda J = \int [\phi(x, y, dx, dy) + \lambda \psi(x, y, dx, dy)]$$

must have its first variation equal to zero.

Let us suppose curves

$$f(x, y, c_1, c_2, \lambda) = 0 \tag{6}$$

found which satisfy condition (5). The equation contains three arbitrary constants, two of which may in general be used to pass the curve through the two fixed points A and B, and the third may be so determined that the integral J takes the prescribed value a.

We then have a curve which satisfies equation (5) and gives J the required value. If the curve is then so varied that J does not change, condition (4) is satisfied and hence, by virtue of (5), condition (3) is fulfilled. The problem is therefore solved.

As an example, let us find the curve of given length which will inclose the maximum area.

We have to make $A = \frac{1}{2}\int (x\,dy - y\,dx)$

a maximum while keeping

$$s = \int \sqrt{dx^2 + dy^2} = a,$$

where a is a given constant. Without loss of generality we assume that the curve starts and returns to O and is tangent to OX at that point. We consider the integral

$$\int \left[\tfrac{1}{2}(x\,dy - y\,dx) + \lambda\sqrt{dx^2 + dy^2}\right]$$

and, applying to it equation (8), § 129, have

$$\frac{1}{2}\,dy - d\left(-\frac{y}{2} + \frac{\lambda\,dx}{\sqrt{dx^2 + dy^2}}\right) = 0;$$

whence $$\frac{dy}{dx} = \frac{\sqrt{\lambda^2 - (y - c_1)^2}}{y - c_1},$$

the solution of which is $(x - c_1)^2 + (y - c_2)^2 = \lambda^2.$

By the conditions imposed upon the curve, $c_1 = 0$ and $\lambda = \pm c_2$. The solution is the circle

$$x^2 + (y - c_2)^2 = c_2^2,$$

and c_2 can obviously be so determined that the circle has the required length.

132. Any number of variables. The discussion and results of §§128 to 131 are easily extended to any number of variables. If we have the integral

$$I = \int f(x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n')dt, \tag{1}$$

where $x_1' = \frac{dx_1}{dt}$, $x_2' = \frac{dx_2}{dt}$, $\cdots$, $x_n' = \frac{dx_n}{dt}$, we have, by the methods of §128, as the differential equations of the curves for which $\delta I = 0$,

$$\begin{aligned} \frac{\partial f}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial f}{\partial x_1'}\right) &= 0, \\ \frac{\partial f}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial f}{\partial x_2'}\right) &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \\ \frac{\partial f}{\partial x_n} - \frac{d}{dt}\left(\frac{\partial f}{\partial x_n'}\right) &= 0. \end{aligned} \tag{2}$$

If the integral is written as

$$I = \int \phi(x_1, x_2, \cdots, x_n, dx_1, dx_2, \cdots, dx_n), \tag{3}$$

the equations are

$$\begin{aligned} \frac{\partial \phi}{\partial x_1} - d\left(\frac{\partial \phi}{\partial (dx_1)}\right) &= 0, \\ \frac{\partial \phi}{\partial x_2} - d\left(\frac{\partial \phi}{\partial (dx_2)}\right) &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \\ \frac{\partial \phi}{\partial x_n} - d\left(\frac{\partial \phi}{\partial (dx_n)}\right) &= 0, \end{aligned} \tag{4}$$

of which one is in general superfluous. The conditions for variable limits and constrained maxima and minima are sufficiently obvious from §§130 and 131.

In addition to the problem of constrained maxima and minima, as discussed in §131, we may have the problem of rendering the integral (3) a maximum or a minimum under the condition that the variables are connected by the condition

$$F(x_1, x_2, \cdots, x_n) = 0. \tag{5}$$

The variations δx_i are then bound by the condition obtained from (5),

$$\frac{\partial F}{\partial x_1}\delta x_1 + \frac{\partial F}{\partial x_2}\delta x_2 + \cdots + \frac{\partial F}{\partial x_n}\delta x_n = 0; \tag{6}$$

and, in addition, we have, as in § 129,

$$\left[\frac{\partial \phi}{\partial x_1} - d\left(\frac{\partial \phi}{\partial (dx_1)}\right)\right]\delta x_1 + \cdots + \left[\frac{\partial \phi}{\partial x_n} - d\left(\frac{\partial \phi}{\partial (dx_n)}\right)\right]\delta x_n = 0 \tag{7}$$

for all variations consistent with (6). From (6) and (7), using λ as an undetermined multiplier, we have

$$\sum_{i=0}^{i=n}\left[\frac{\partial \phi}{\partial x_i} - d\left(\frac{\partial \phi}{\partial (dx_i)}\right) + \lambda\frac{\partial F}{\partial x_i}\right]\delta x_i = 0. \tag{8}$$

Hence if λ is so determined that the coefficient of one of the variations δx_i in (8) vanishes, the other coefficients must also vanish, since in (6) all but one of the variations are arbitrary. Therefore the equations to determine λ and the required curve are

$$\frac{\partial \phi}{\partial x_i} - d\left(\frac{\partial \phi}{\partial (dx_i)}\right) + \lambda\frac{\partial F}{\partial x_i} = 0. \qquad (i = 1, 2, \cdots, n) \tag{9}$$

For example, let it be required to find the shortest lines (geodesics) on any surface

$$F(x, y, z) = 0. \tag{10}$$

This is to minimize the integral

$$s = \int \sqrt{dx^2 + dy^2 + dz^2} \tag{11}$$

subject to the condition (10). Using formulas (9), we have

$$\begin{aligned} -d\frac{dx}{\sqrt{dx^2 + dy^2 + dz^2}} + \lambda\frac{\partial F}{\partial x} &= 0, \\ -d\frac{dy}{\sqrt{dx^2 + dy^2 + dz^2}} + \lambda\frac{\partial F}{\partial y} &= 0, \\ \text{and} \qquad -d\frac{dz}{\sqrt{dx^2 + dy^2 + dz^2}} + \lambda\frac{dF}{\partial z} &= 0. \end{aligned} \tag{12}$$

We may draw from equations (12) the result

$$d\left(\frac{dx}{ds}\right) : d\left(\frac{dy}{ds}\right) : d\left(\frac{dz}{ds}\right) = \frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z},$$

or

$$\frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2} = \frac{\partial F}{\partial x} : \frac{\partial F}{\partial y} : \frac{\partial F}{\partial z};$$

whence we infer the important theorem that the principal normal of a geodesic coincides with the normal to the surface.

133. Hamilton's principle; Lagrange's equations. Consider a particle of mass m_i moving along a curve under the influence of a force whose components are X_i, Y_i, Z_i. Then the path is determined by the equations

$$m_i\frac{d^2x_i}{dt^2} = X_i, \qquad m_i\frac{d^2y_i}{dt^2} = Y_i, \qquad m_i\frac{d^2z_i}{dt^2} = Z_i. \tag{1}$$

Let the curve be varied without changing its extremities and let δx_i, δy_i, δz_i be the variations of x_i, y_i, z_i respectively. By virtue of (1),

$$\left(m_i\frac{d^2x_i}{dt^2} - X_i\right)\delta x_i + \left(m_i\frac{d^2y_i}{dt^2} - Y_i\right)\delta y_i + \left(m_i\frac{d^2z_i}{dt^2} - Z_i\right)\delta z_i = 0. \tag{2}$$

This is true for each of the particles of a system, and, summing over the whole number of particles, we have

$$\sum m_i\left(\frac{d^2x_i}{dt^2}\delta x_i + \frac{d^2y_i}{dt^2}\delta y_i + \frac{d^2z_i}{dt^2}\delta z_i\right) = \sum(X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i). \tag{3}$$

The work done by this displacement is

$$\delta W = \sum(X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i). \tag{4}$$

The kinetic energy of the system is

$$T = \frac{1}{2}\sum m_i\left[\left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz_i}{dt}\right)^2\right],$$

and the variation of T is

$$\begin{aligned}\delta T &= \sum m_i\left[\frac{dx_i}{dt}\,\delta\left(\frac{dx_i}{dt}\right) + \frac{dy_i}{dt}\,\delta\left(\frac{dy_i}{dt}\right) + \frac{dz_i}{dt}\,\delta\left(\frac{dz_i}{dt}\right)\right]\\ &= \sum m_i\left[\frac{dx_i}{dt}\frac{d}{dt}(\delta x_i) + \frac{dy_i}{dt}\frac{d}{dt}(\delta y_i) + \frac{dz_i}{dt}\frac{d}{dt}(\delta z_i)\right],\end{aligned} \tag{5}$$

the last change being made by (7), § 128.

Now we have

$$\frac{d}{dt}\left[\frac{dx_i}{dt}\,\delta x_i+\frac{dy_i}{dt}\,\delta y_i+\frac{dz_i}{dt}\,\delta z_i\right]=\left[\frac{d^2x_i}{dt^2}\,\delta x_i+\frac{d^2y_i}{dt^2}\,\delta y_i+\frac{d^2z_i}{dt^2}\,\delta z_i\right]$$
$$+\frac{dx_i}{dt}\frac{d}{dt}(\delta x_i)+\frac{dy_i}{dt}\frac{d}{dt}(\delta y_i)+\frac{dz_i}{dt}\frac{d}{dt}(\delta z_i), \tag{6}$$

so that, by virtue of (4), (5), and (6), equation (3) may be written

$$\sum m_i\frac{d}{dt}\left[\frac{dx_i}{dt}\,\delta x_i+\frac{dy_i}{dt}\,\delta y_i+\frac{dz_i}{dt}\,\delta z_i\right]=\delta T+\delta W. \tag{7}$$

By multiplying (7) by dt and integrating between the times $t=t_0$ and $t=t_1$, at which the body is at the beginning and end of its path, we have

$$\int_{t_0}^{t_1}(\delta T+\delta W)dt=\sum m_i\left[\frac{dx_i}{dt}\,\delta x_i+\frac{dy_i}{dt}\,\delta y_i+\frac{dz_i}{dt}\,\delta z_i\right]_{t_0}^{t_1}=0, \tag{8}$$

where the right-hand member is zero, since by hypothesis δx_i, δy_i, δz_i are zero at the beginning and end of the path.

We shall now assume that there exists a potential energy V such that

$$W=-V.$$

Then equation (8) may be written

$$\delta\int_{t_0}^{t_1}(T-V)dt=0; \tag{9}$$

that is, *the body so moves that the time integral of the difference between its kinetic and potential energies has a first variation zero.* This is *Hamilton's principle.*

The position of the body may be determined by n parameters $q_1, q_2, \cdots, q_n$, sometimes called generalized coördinates. Then x_i, y_i, z_i depend only on q_i, and $\frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}$ depend on q_i and $\dot{q}_i$, where $\dot{q}_i=\frac{dq_i}{dt}$, so that T in (9) is a function of q_i and $\dot{q}_i$, and V is a function of q_i.

We can now apply formulas (2), §132, to (9). We have the n equations

$$\frac{\partial(T-V)}{\partial q_i}-\frac{d}{dt}\left[\frac{\partial(T-V)}{\partial\dot{q}_i}\right]=0. \qquad (i=1, 2, \cdots, n) \tag{10}$$

These are *Lagrange's equations* for the motion of the system.

For example, consider the motion of a pendulum of any form swinging in a plane about a point of suspension O. The position of the pendulum is fully determined by the angle θ which the line from O to the center of gravity of the pendulum makes with the vertical. We have then only one parameter q_i, namely θ.

We have, by mechanics,

$$T = \tfrac{1}{2} I\dot{\theta}^2,$$
$$V = Mgh(1 - \cos\theta),$$

where I is the moment of inertia of the pendulum about O, h the distance from O to the center of gravity of the pendulum, and M the mass of the pendulum. Then (10) becomes

$$- Mgh \sin\theta - \frac{d}{dt}(I\dot{\theta}^2) = 0,$$

or

$$I\frac{d^2\theta}{dt^2} = - Mgh \sin\theta.$$

EXERCISES

1. Find the equation of a straight line in Cartesian coördinates by minimizing the integral $$\int\sqrt{dx^2 + dy^2}.$$

2. Find the equation of a straight line in polar coördinates by minimizing the integral $$\int\sqrt{dr^2 + r^2\,d\theta^2}.$$

3. Find the equation of the shortest line on the surface of a sphere and prove that it is a great circle.

4. Show that the shortest lines on a right circular cylinder are helices.

5. Find the equation of the shortest line on a cone of revolution.

6. Find the equation of the shortest line on the helicoid $x = r\cos\theta$, $y = r\sin\theta$, $z = k\theta$.

7. Find the differential equation of a geodesic on any surface of revolution $x = r\cos\theta$, $y = r\sin\theta$, $z = f(r)$.

8. Given that the velocity of a body sliding from rest along a curve is $\sqrt{2gh}$, where h is the vertical distance of the fall, find the equation of the brachistochrone; that is, the curve in which the body falls in least time from a point O to a point B.

9. Determine the curve for which $\int v\,ds$ is a minimum, assuming that the velocity $v = \sqrt{2g(y + a)}$.

10. Find the curve of given length between two fixed points which generates the minimum surface of revolution.

11. Find a curve of given length between two fixed points such that the area bounded by it, the axis of x, and two ordinates is a maximum.

12. Find a curve of given length between two fixed points such that the area bounded by it and two lines to the origin is a maximum.

13. Assuming that a string with fixed ends will so hang that its center of gravity has the lowest possible position, find the curve in which a string of given length will hang.

14. Prove by Hamilton's principle that the string in Ex. 13 will hang as stated.

15. Show by Hamilton's principle that the equations in polar coördinates for the motion of a particle in a plane are

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = f_r,$$

$$m\left[r\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dr}{dt}\right] = f_\theta,$$

where f_r and f_θ are the components of force acting on the particle along the radius vector and normal to it.

16. Find from Hamilton's principle the polar equations for the motion of a particle in space.

17. Show that if the integral

$$\iint F(x, y, z, p, q)dx\,dy,$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, is made a maximum or a minimum by a surface spanned in a given closed curve, that surface must satisfy the equation

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial p} - \frac{d}{dy}\frac{\partial F}{\partial q} = 0.$$

18. Use Ex. 17 to show that a minimum surface, that is, a surface of least area in a given contour, satisfies the partial differential equation

$$r(1 + q^2) - 2\,pqs + t(1 + p^2) = 0,$$

where $$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x\,\partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

19. Show that the only surface of revolution which satisfies the equation in Ex. 18 is that formed by revolving a catenary (except for the trivial case of a plane formed by revolving a straight line about an axis perpendicular to it).

CHAPTER XV

FUNCTIONS OF A COMPLEX VARIABLE

134. Complex numbers. A complex number is a quantity of the form

$$x + yi, \tag{1}$$

where x and y are real numbers and i is a unit defined by the equation

$$i = \sqrt{-1}. \tag{2}$$

The number x is the real part of the complex number and the number yi is the imaginary part. When $y = 0$ the complex number becomes a real number, so that the real numbers form a subclass of the complex numbers; when $x = 0$ the complex number becomes a pure imaginary number.

The complex numbers being thus defined, it is necessary to lay down rules for their manipulation. These are essentially two:

I. A complex number (1) *is zero when, and only when,* $x = 0$ *and* $y = 0$.

II. The complex numbers obey the ordinary laws of algebra, with the addition that $i = \sqrt{-1}$.

From these follow at once the formulas for addition, subtraction, and multiplication; namely,

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2), \tag{3}$$

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1). \tag{4}$$

The quotient of two numbers such as

$$\frac{x_1 + iy_1}{x_2 + iy_2}$$

may be most conveniently found by multiplying dividend and divisor by $x_2 - iy_2$, thus,

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \tag{5}$$

From (3), (4), and (5) we have the following theorem:

III. The sum, difference, product, and quotient of two complex numbers are themselves complex numbers.

If two complex numbers are equal, that is, if

$$x_1 + iy_1 = x_2 + iy_2,$$

then, from (3), $$(x_1 - x_2) + i(y_1 - y_2) = 0;$$

whence, from postulate I,

$$x_1 = x_2, \qquad y_1 = y_2.$$

That is,

IV. Two complex numbers are equal when, and only when, the real part of one is equal to the real part of the other and the pure imaginary part of one is equal to the pure imaginary part of the other.

Two quantities which differ only in the sign of their pure imaginary parts are called *conjugate imaginary.* Thus $a + bi$ and $a - bi$ are conjugate imaginary.

135. Graphical representation and trigonometric form. Complex numbers are essentially algebraic quantities, but they may be given a convenient geometric representation.

Construct axes of coördinates OX and OY (Fig. 105) and take any point P. Then to any point P corresponds a definite pair of values (x, y), and conversely. Therefore to P may be made to correspond the complex number z, where

$$z = x + iy. \tag{1}$$

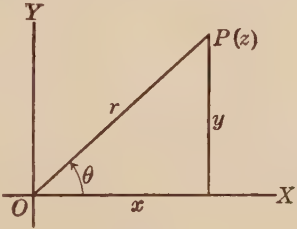

FIG. 105

In this connection OX is called the axis of reals, since real numbers are represented by points upon it, and OY is called the axis of imaginaries.

If we introduce polar coördinates

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

we then have, from (1),

$$z = r(\cos\theta + i\sin\theta), \tag{2}$$

which is the trigonometric form in which a complex number can always be put.

The number r, which is always taken positive, is called the *modulus*, or the *absolute value*, of z and is equal to the length of the line OP. Then

$$|z| = r = \sqrt{x^2 + y^2}. \tag{3}$$

The angle θ is called the *angle*, or *argument*, of z. Then

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}. \tag{4}$$

If a is a real positive number, we may write

$$\begin{aligned} a &= a(\cos 0 + i\sin 0),\\ -a &= a(\cos\pi + i\sin\pi),\\ ai &= a\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right),\\ -ai &= a\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right), \end{aligned}$$

thus expressing any real or pure imaginary number in the general form (2) and exhibiting the modulus and the angle of each.

Any multiple of 2π may be added to the angle θ without altering z, since

$$r[\cos(\theta+2k\pi) + i\sin(\theta+2k\pi)] = r(\cos\theta + i\sin\theta), \tag{5}$$

where k is any integer.

Take two complex quantities z_1 and z_2, represented by the points P_1 and P_2 (Fig. 106) respectively. It is easy to see from (3), § 134, that their sum, z_1+z_2, is represented by the point P_3, found by constructing a parallelogram on the sides OP_1 and OP_2. From the figure it follows that

$$|z_1+z_2| \leqq |z_1|+|z_2|, \tag{6}$$

the equality sign holding only when OP_1 and OP_2 are in the same straight line.

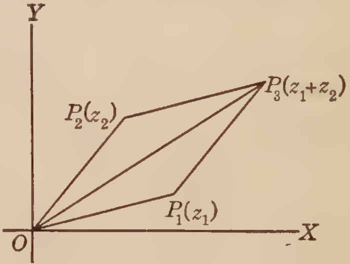

FIG. 106

Since $z_1 - z_2 = z_1 + (-z_2)$ and $|-z_2| = |z_2|$, we have also, from (6),

$$|z_1 - z_2| \leqq |z_1|+|z_2|. \tag{7}$$

Graphically the points z_2 and $-z_2$ are symmetrically placed with respect to the origin, and we have for subtraction Fig. 107.

To represent multiplication and division graphically we use the trigonometric form. Then

$$\begin{aligned} z_1z_2 &= r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\\ &\qquad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]\\ &= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \end{aligned} \tag{8}$$

Hence in the multiplication of two complex numbers the moduli are multiplied and the angles are added. Graphically, if P_1 (Fig. 108) is the point z_1, the point z_1z_2 may be found by rotating OP_1 in the positive direction through an angle θ_2 and stretching or contracting OP_1 until it is of length r_1r_2. In particular, multiplication by i may be considered as rotating OP_1 through an angle of 90°, multiplication by -1 as rotation through an angle of 180°, and multiplication by $-i$ as rotation through 270°.

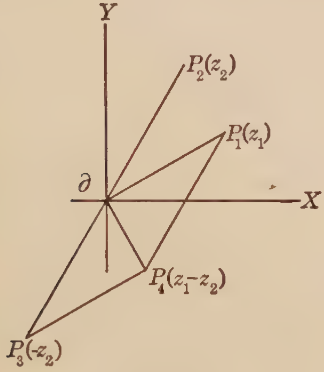

FIG. 107

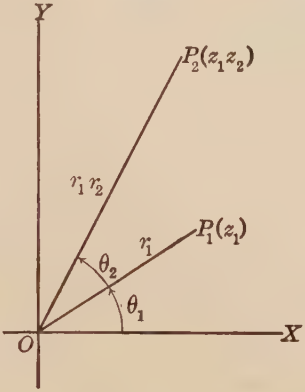

FIG. 108

For division we have

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \quad (9)$$

Hence in dividing one complex quantity by another the angle of the divisor is subtracted from the angle of the dividend, and the modulus of the dividend is divided by the modulus of the divisor. Graphically the line OP_1 is rotated in a negative direction through an angle θ_2, and the length of OP_1 is divided by r_2.

From (8) and (9) we have

$$|z_1z_2| = |z_1|\cdot|z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \quad (10)$$

136. Powers and roots. The value of z^n, where n is a positive integer, may be found by successive multiplication of z by itself. If we write z^n as $x + iy$, we can find $(x + iy)^n$ by applying the binomial theorem; thus,

$$(x + iy)^2 = x^2 - y^2 + 2\,xyi,$$
$$(x + iy)^3 = x^3 - 3\,xy^2 + i(3\,x^2y - y^3).$$

An important form of the power is obtained by using the trigonometric form of z. From the previous section,

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta). \tag{1}$$

In particular, if $r = 1$ we have De Moivre's theorem (§ 26)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{2}$$

The root $z^{\frac{1}{n}}$, where n is a positive integer, is a number which raised to the nth power gives z. From the general form of z as given in (5), § 135, it is evident from (1) that

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right), \tag{3}$$

and we shall get n distinct values of $z^{\frac{1}{n}}$ by giving to k the values $0, 1, 2, \cdots, (n-1)$ successively. In this work $r^{\frac{1}{n}}$ is to be taken as the numerical positive root of the real positive number r.

By combining (1) and (3) we have

$$z^{\frac{p}{q}} = r^{\frac{p}{q}}\left(\cos\frac{p\theta + 2kp\pi}{q} + i\sin\frac{p\theta + 2kp\pi}{q}\right), \tag{4}$$

where $k = 0, 1, 2, \cdots, (q-1)$.

Finally,
$$z^{-m} = \frac{1}{z^m} = \frac{\cos 0 + i\sin 0}{r^m(\cos m\theta + i\sin m\theta)}$$
$$= r^{-m}[\cos(-m\theta) + i\sin(-m\theta)]. \tag{5}$$

Hence formula (1) is true for any rational value of n.

We may now prove the relation which was used in obtaining formula (5), § 70. By algebra,

$$x^{2p} - 1 = (x - r_1)(x - r_2)\cdots(x - r_{2p}), \tag{6}$$

where $r_1, r_2, \cdots, r_{2p}$ are the roots of the equation $x^{2p} = 1$. These roots are, by (3),

$$\cos\frac{2k\pi}{2p} + i\sin\frac{2k\pi}{2p}. \quad (k = 0, 1, \cdots, 2p-1) \tag{7}$$

When $k = 0$, (7) gives the root $r_1 = 1$; when $k = p$, (7) gives the root $r_p = -1$. The other roots pair off into conjugate imaginary pairs. For when $k = n$, where $n = 1, 2, \cdots, p-1$,

$$r_n = \cos\frac{n\pi}{p} + i\sin\frac{n\pi}{p},$$

and when $k = 2p - n$,

$$r_{2p-n} = \cos\frac{n\pi}{p} - i\sin\frac{n\pi}{p}.$$

Therefore $(x - r_n)(x - r_{2p-n}) = x^2 - 2x\cos\frac{n\pi}{p} + 1$. Hence we may write (6) in the form

$$\frac{x^{2p}-1}{x^2-1} = \left(x^2 - 2x\cos\frac{\pi}{p} + 1\right)\left(x^2 - 2x\cos\frac{2\pi}{p} + 1\right)\cdots \left(x^2 - 2x\cos\frac{p-1}{p}\pi + 1\right). \tag{8}$$

Now let $x \to 1$. The left-hand side of (8) approaches p, and the limit of the right side is its value when $x = 1$. Therefore,

$$p = 2^{2p-2}\sin^2\frac{\pi}{2p}\sin^2\frac{2\pi}{2p}\cdots\sin^2\frac{(p-1)\pi}{2p}.$$

Similarly, let $x \to -1$. Then

$$p = 2^{2p-2}\cos^2\frac{\pi}{2p}\cos^2\frac{2\pi}{2p}\cdots\cos^2\frac{(p-1)\pi}{2p}.$$

Multiply the last two results, using the formula for the double angle, and take the square root. We then have

$$p = 2^{p-1}\sin\frac{\pi}{p}\sin\frac{2\pi}{p}\cdots\sin\frac{(p-1)\pi}{p};$$

whence
$$\sin\frac{\pi}{p}\sin\frac{2\pi}{p}\cdots\sin\frac{(p-1)\pi}{p} = \frac{p}{2^{p-1}}.$$

137. The square root. Let us consider in detail the dependence of $w = \sqrt{z}$ on the value of z. From (3), §136, there are two values of w; namely,

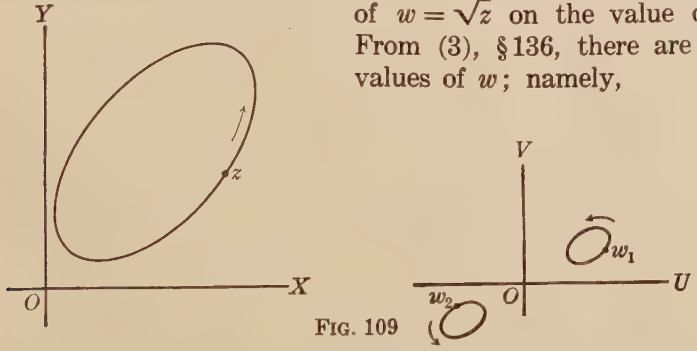

FIG. 109

$$w_1 = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right),$$

and
$$w_2 = \sqrt{r}\left(\cos\frac{\theta+2\pi}{2} + i\sin\frac{\theta+2\pi}{2}\right) = -w_1.$$

We may plot z on the (x, y) plane and w on the (u, v) plane, where $w = u + iv$ (Fig. 109).

Let z describe a curve in its plane. Then w_1 and w_2 each describes a curve, and the two curves do not intersect unless $z = 0$.

If z describes a closed curve which does not go around O, θ returns to its original value, and w_1 and w_2 return each to its original value.

If z goes around O, θ changes from θ_0 (its original value) to $\theta_0 + 2\pi$, w_1 becomes w_2, and w_2 becomes w_1 (Fig. 110). To make this numerically clear we give a table of values of w_1 corresponding to successive values of z:

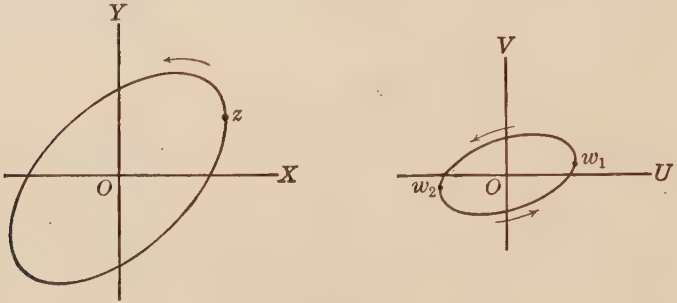

FIG. 110

$$z = 4, \qquad w_1 = 2.$$

$$z = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 4\,i, \qquad w_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} + i\,\frac{2}{\sqrt{2}}.$$

$$z = 4(\cos\pi + i\sin\pi) = -4, \qquad w_1 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2\,i.$$

$$z = 4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -4\,i, \quad w_1 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\frac{2}{\sqrt{2}} + i\,\frac{2}{\sqrt{2}}.$$

$$z = 4(\cos 2\pi + i\sin 2\pi) = 4, \qquad w_1 = 2(\cos\pi + i\sin\pi) = -2.$$

It is evident that by a passage of z around the origin, $\sqrt{z}$ changes its sign. Similarly, by a passage of z around $z = a$, the radical $\sqrt{z-a}$ or $\sqrt{a-z}$ changes its sign. Of course, an even number of circuits around $z = a$ leaves $\sqrt{z-a}$ unchanged, and an odd number changes its sign.

Consider
$$\sqrt{1-z^2} = \sqrt{1-z}\,\sqrt{1+z}.$$

A single circuit around either of the points $+1$ or -1 changes the sign of $\sqrt{1-z^2}$; a circuit about both of them leaves the sign unchanged.

Now let z describe a path as follows: Let it start from O (Fig. 111), go along the axis of reals to $1-r$, describe then the small semicircle around 1 to $1+r$, and then proceed indefinitely along the axis of reals. What is the effect on the sign of $\sqrt{1-z^2}$ if at the outset $\sqrt{1-z^2}=+1$? To answer this question we place

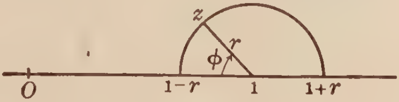

FIG. 111

$$1-z=1-x-iy=r(\cos\phi-i\sin\phi),$$

where ϕ and r are as shown in Fig. 111. Then

$$1+z=2-r(\cos\phi-i\sin\phi),$$

and $$\sqrt{1-z^2}=r^{\frac{1}{2}}\left(\cos\frac{\phi}{2}-i\sin\frac{\phi}{2}\right)\sqrt{2-r(\cos\phi-i\sin\phi)}. \qquad (1)$$

When z is between 0 and $1-r$, $\phi=0$, and we have, from (1),

$$\sqrt{1-z^2}=r^{\frac{1}{2}}\sqrt{2-r},$$

which must be taken as $+$, since by hypothesis $\sqrt{1-z^2}=1$ when $r=0$.

When z is real and $>1+r$, $\phi=\pi$, and we have, from (1),

$$\sqrt{1-z^2}=-ir^{\frac{1}{2}}\sqrt{2+r}. \qquad (2)$$

If we simply put $z=1+r$ in $\sqrt{1-z^2}$, we get $\sqrt{-2r-r^2}$, and the analysis just given shows that this must be taken as $-i\sqrt{2r+r^2}$ and not as $i\sqrt{2r+r^2}$.

138. Exponential and trigonometric functions. By definition we have

$$e^z=1+\frac{z}{1}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots, \qquad (1)$$

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots, \qquad (2)$$

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots. \qquad (3)$$

When z is real these become the elementary functions. To prove the convergence of the series place $z=r(\cos\theta+i\sin\theta)$. Then (1) becomes

$$\left(1+r\cos\theta+\frac{r^2}{2!}\cos 2\theta+\cdots\right)+i\left(r\sin\theta+\frac{r^2}{2!}\sin 2\theta+\cdots\right).$$

Each term of the two series in parentheses is at most equal to a corresponding term of the known convergent series

$$1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \cdots,$$

and hence by the comparison test (§ 18) the series converges. In the same manner (2) and (3) may be shown to converge.

From (1) we have

$$e^0 = 1, \qquad e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}, \tag{4}$$

which are the fundamental properties of the exponential function.

From (1), (2), and (3) we get also

$$\begin{aligned} e^{iz} &= \cos z + i \sin z, \\ e^{-iz} &= \cos z - i \sin z, \end{aligned} \tag{5}$$

which is true for all complex values of z. From (4) and (5) it follows that

$$e^{x+iy} = e^x \cos y + ie^x \sin y, \tag{6}$$

from which we have the theorem:

The exponential function of a complex quantity is itself a complex quantity.

From (5) we have

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{7}$$

With the aid of (4) we readily obtain from (7)

$$\sin (z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \tag{8}$$

$$\cos (z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2. \tag{9}$$

In (7) let us place $z_1 = x$, $z_2 = iy$. We get

$$\begin{aligned} \sin (x + iy) &= \sin x \cos iy + \cos x \sin iy \\ &= \frac{e^y + e^{-y}}{2} \sin x + i \frac{e^y - e^{-y}}{2} \cos x \\ &= \cosh y \sin x + i \sinh y \cos x. \end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned} \cos (x + iy) &= \frac{e^y + e^{-y}}{2} \cos x - i \frac{e^y - e^{-y}}{2} \sin x \\ &= \cosh y \cos x - i \sinh y \sin x; \end{aligned} \tag{11}$$

whence we have the theorem:

The sine and cosine of a complex quantity are themselves complex quantities.

It is known from elementary trigonometry that $\sin(x + 2k\pi) = \sin x$, $\cos(x + 2k\pi) = \cos x$, where x is a real number and k is an integer. Hence, from (6),

$$e^{z+2k\pi i} = e^z, \tag{12}$$

and, from (10) and (11),

$$\sin(z + 2k\pi) = \sin z, \qquad \cos(z + 2k\pi) = \cos z. \tag{13}$$

From this we have the theorem:

The exponential function is a periodic function with the imaginary period $2\pi i$. The sine and cosine are periodic functions with the real period 2π.

139. The hyperbolic functions. The hyperbolic sine and the hyperbolic cosine have been defined in § 27 and treated for a real variable. The same definitions hold for a complex variable; namely,

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{1}$$

From these definitions we have

$$\sinh iz = \frac{e^{iz} - e^{-iz}}{2} = i \sin z, \tag{2}$$

$$\cosh iz = \frac{e^{iz} + e^{-iz}}{2} = \cos z. \tag{3}$$

Similarly,

$$\sinh z = \frac{e^{-i(iz)} - e^{i(iz)}}{2} = -i \sin iz, \tag{4}$$

$$\cosh z = \frac{e^{-i(iz)} + e^{i(iz)}}{2} = \cos iz. \tag{5}$$

From this it appears that hyperbolic functions are essentially trigonometric functions and that relations between trigonometric functions give rise to relations between hyperbolic functions, with certain differences arising from the presence of the factor i in (2) and (3).

For example, since $\sin^2 iz + \cos^2 iz = 1$,

we have, from (2) and (3),

$$\cosh^2 z - \sinh^2 z = 1, \tag{6}$$

which may be verified from (1).

We have also

$$\begin{aligned}\sinh(z_1+z_2) &= -i\sin(iz_1+iz_2)\\ &= -i\sin iz_1\cos iz_2 - i\cos iz_1\sin iz_2\\ &= \sinh z_1\cosh z_2+\cosh z_1\sinh z_2, \qquad (7)\\ \cosh(z_1+z_2) &= \cos(iz_1+iz_2)\\ &= \cos iz_1\cos iz_2-\sin iz_1\sin iz_2\\ &= \cosh z_1\cosh z_2+\sinh z_1\sinh z_2. \qquad (8)\end{aligned}$$

As special cases of (7) and (8) we have, by use of (2) and (3),

$$\begin{aligned}\sinh(x+iy) &= \sinh x\cosh iy+\cosh x\sinh iy\\ &= \sinh x\cos y+i\cosh x\sin y, \qquad (9)\\ \cosh(x+iy) &= \cosh x\cosh iy+\sinh x\sinh iy\\ &= \cosh x\cos y+i\sinh x\sin y, \qquad (10)\end{aligned}$$

by means of which the hyperbolic sine and the hyperbolic cosine are separated into their real and imaginary parts.

From (7) and (8) we have

$$\begin{aligned}\sinh(z+2k\pi i) &= \sinh z,\\ \cosh(z+2k\pi i) &= \cosh z.\end{aligned}$$

Hence

The hyperbolic sine and the hyperbolic cosine are periodic functions with the imaginary period $2\pi i$.

140. The logarithmic function. If $z=e^w$, then, by definition,

$$w=\log z.$$

The properties of the logarithmic function, namely,

$$\log(z_1z_2)=\log z_1+\log z_2, \qquad \log\frac{z_1}{z_2}=\log z_1-\log z_2,$$

$$\log z^n=n\log z, \qquad \log 1=0,$$

are deduced from the definition, as in the case of real variables.

The logarithm of a complex number is itself a complex number. For let us place

$$z=x+iy=r(\cos\theta+i\sin\theta)=re^{i\theta};$$

then

$$\log z=\log r+\log e^{i\theta}=\log r+i\theta=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}.$$

Here $\log r$ is the real logarithm of the positive number r, as found in the usual tables.

We may now find the logarithm of a real negative number. For, if $-a$ is such a number, we may write $-a = a(\cos \pi + i \sin \pi) = ae^{i\pi}$; whence

$$\log(-a) = \log a + i\pi.$$

In particular, $$\log(-1) = i\pi.$$

It is to be noticed that in the domain of the complex numbers a logarithm is not a unique quantity. For

$$e^w = e^{w+2ki\pi} = z,$$

where k is zero or an integer. Therefore

$$\log z = w + 2\,ki\pi.$$

From this it follows that the logarithm of any number has an infinite number of values differing by multiples of $2\,\pi i$.

Let us consider the effect upon $w = \log z$ by varying z continuously. When $z = z_1$, let us pick any one of the possible values of w, say

$$w_1 = \log r_1 + i\theta_1.$$

When z describes a path from z_1 back to z_1 without going around the origin, r_1 and θ_1 return to their original values, and hence w_1 returns to its original value. But if z describes a path which goes around O once in a positive direction, r_1 returns to its original value, but the angle θ_1 becomes $\theta_1 + 2\,\pi$, and hence w_1 becomes $w_1 + 2\,\pi i$. If w goes m times around the origin in a positive direction and n times in a negative direction, w_1 becomes $w_1 + 2(m-n)\pi i$.

141. The inverse hyperbolic and trigonometric functions. If

$$z = \sinh w,$$

then, by definition, $$w = \sinh^{-1} z;$$

if $$z = \cosh w,$$

then $$w = \cosh^{-1} z;$$

and if $$z = \tanh w,$$

then $$w = \tanh^{-1} z.$$

These functions are closely connected with the logarithms. For let

$$z = \sinh w = \frac{e^w - e^{-w}}{2}.$$

From this equation we have

$$e^{2w} - 2ze^{w} - 1 = 0,$$

which is a quadratic equation in e^w.

Hence $$e^w = z \pm \sqrt{z^2 + 1},$$

$$w = \sinh^{-1} z = \log\left(z \pm \sqrt{z^2 + 1}\right). \tag{1}$$

Similarly, $$\cosh^{-1} z = \log\left(z \pm \sqrt{z^2 - 1}\right), \tag{2}$$

and $$\tanh^{-1} z = \frac{1}{2} \log \frac{1 - z}{1 + z}. \tag{3}$$

These formulas are true for any complex quantity z. If $z = x$, a real number, (1) gives only one real value of w arising from the use of the plus sign in (1). If $z = x$, a real number, in (2), we have two real values of w provided $x > 1$. If $z = x$, a real number, in (3), we have one real value of w when $-1 < x < 1$.

The functions $\sin^{-1} z$, $\cos^{-1} z$, and $\tan^{-1} z$ may also be expressed in terms of logarithms. This may be done in the same manner as that just employed for the hyperbolic functions, or we may work as follows:

Let $$z = \sin w = \frac{1}{i} \sinh iw.$$

Then $$w = \sin^{-1} z = \frac{1}{i} \sinh^{-1} iz$$

$$= \frac{1}{i} \log\left(iz \pm \sqrt{1 - z^2}\right). \tag{4}$$

Let $$z = \cos w = \cosh iw.$$

Then $$w = \cos^{-1} z = \frac{1}{i} \cosh^{-1} z$$

$$= \frac{1}{i} \log\left(z \pm \sqrt{z^2 - 1}\right). \tag{5}$$

Let $$z = \tan w = \frac{1}{i} \tanh iw.$$

Then $$w = \tan^{-1} z = \frac{1}{i} \tanh^{-1} iz$$

$$= \frac{1}{2i} \log \frac{1 - iz}{1 + iz}. \tag{6}$$

142. Functions of a complex variable in general. We have seen that functions of a complex variable obtained by operating on $x + iy$ with the fundamental operations of algebra, or involving the elementary transcendental functions, are themselves complex numbers of the form $u + iv$, where u and v are real functions of x and y. Let us now assume the expression $w = u + iv$, and inquire what conditions it must satisfy in order that it may be a function of $z = x + iy$.

In the first place, it is to be noticed that in the broadest sense of the word *function* (§ 1) w is always a function of z, since when z is given, x and y are determined and therefore u and v are determined. But this definition is too broad for our present purpose, and we shall restrict it by demanding that the function shall have a definite derivative for a definite value of z.

In case $f(z)$ is given explicitly in z involving the elementary functions of the previous sections, this condition is surely met, because all the operations used in the calculus of a real variable to obtain the elementary derivatives are valid for the complex variable, and the derivative is uniquely determined. We have, for example,

$$\frac{d}{dz} z^n = nz^{n-1}, \quad \frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \log z = \frac{1}{z},$$

and so on.

We shall proceed to show, however, that the uniqueness of the derivative means that u and v satisfy certain conditions. This we do as follows: In order to obtain an increment of z, we may assign at pleasure increments Δx and Δy to x and y, respectively, and obtain $\Delta z = \Delta x + i\,\Delta y$. The direction in which the point Q (Fig. 112), which corresponds to $z + \Delta z$ in the graphical representation, lies from P, which corresponds to z, depends on the ratio $\frac{\Delta y}{\Delta x}$, which may have any value whatever. Corresponding to a given increment Δz, w takes an increment Δw, where, by (1), § 33,

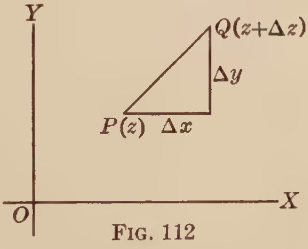

FIG. 112

$$\Delta w = \left(\frac{\partial u}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \epsilon_2\right)\Delta y + i\left[\left(\frac{\partial v}{\partial x} + \epsilon_3\right)\Delta x + \left(\frac{\partial v}{\partial y} + \epsilon_4\right)\Delta y\right],$$

provided u and v have continuous partial derivatives of the first order.

Dividing by $\Delta z = \Delta x + i\,\Delta y$ and taking the limit as $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$, we have

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x} + \left(\dfrac{\partial u}{\partial y} + i\,\dfrac{\partial v}{\partial y}\right)\dfrac{dy}{dx}}{1 + i\,\dfrac{dy}{dx}}. \tag{1}$$

Unless special conditions are imposed upon u and v, the expression on the right-hand side of equation (1) involves $\frac{dy}{dx}$, and the value of $\lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ depends upon the direction in which the point Q approaches the point P. Now the value of the right-hand side of (1) when $\frac{dy}{dx} = 0$ is

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x},$$

and its value when $\frac{dy}{dx} = \infty$ is

$$\frac{1}{i}\left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right).$$

Equating these two values, we have

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}. \tag{2}$$

This, then, is the necessary condition that $\lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ should be the same for the two values $\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = \infty$. It is also the sufficient condition that $\lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ should be the same for all values of $\frac{dy}{dx}$, for if (1) is simplified by aid of (2), $\frac{dy}{dx}$ disappears from it.

Now (2) is equivalent to the two conditions

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned} \tag{3}$$

Hence equations (3) are the necessary and sufficient conditions that the function $u + iv$ should have a derivative with respect to $x+iy$ which depends upon the value of $x+iy$ only. From (1) this derivative may be written

$$\frac{dw}{dz}=f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{1}{i}\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right). \tag{4}$$

A function $u+iv$ which satisfies conditions (3) is called an analytic function of $x+iy$.

143. Conjugate functions. Two real functions u and v, which satisfy conditions (3), § 142, are called conjugate functions. By differentiating the first equation of (3), § 142, with respect to x, the second with respect to y, and adding the results, we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Also, by differentiating the first equation of (3), § 142, with respect to y, the second with respect to x, and taking the difference of the results, we have

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

That is, each of a pair of conjugate functions is a solution of the Laplace differential equation in two variables.

Conversely, any real solution of the Laplace equation may be made the real part of an analytic function $f(z)$.

For let u be such a solution. We may determine v from the equations

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y},\qquad \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}. \tag{1}$$

In fact,
$$dv=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$$

satisfies the condition for an exact differential, since

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right),$$

and v may be found by the method of § 36 or of § 75.

Then the function
$$w+iv=f(z)$$

satisfies the conditions for an analytic function.

Let us now construct the two families of curves $u=c_1$ and $v=c_2$. If (x_1, y_1) is a point of intersection of two of these curves, one from each family, the slopes of the tangent lines at (x_1, y_1) are, respectively,

$$\frac{dy}{dx} = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}}$$

and

$$\frac{dy}{dx} = -\frac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}}.$$

But from (3), § 142, these two slopes are negative reciprocals. Hence the two curves intersect at right angles; that is, every curve of one family intersects every curve of the other family at right angles. We express this by saying that the families of curves corresponding to two conjugate functions form an orthogonal system.

Examples of conjugate functions and of orthogonal systems of curves may be found by taking the real and imaginary parts of any function of a complex variable. We have, for instance,

$$\log(x + iy) = \log\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x}.$$

Hence $\log\sqrt{x^2+y^2}$ and $\tan^{-1}\dfrac{y}{x}$ are conjugate functions, and the curves $x^2 + y^2 = c_1$ and $y = c_2 x$ form an orthogonal system. In fact, one family of curves consists of circles with their centers at the origin, and the other consists of straight lines through the origin.

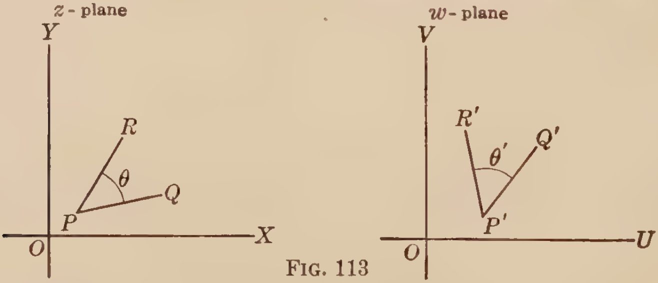

Fig. 113

144. Conformal representation. An equation

$$w = f(z), \tag{1}$$

where $z = x + iy$, $w = u + iv$, and $f(z)$ is an analytic function, establishes a relation between the plane in which z is represented as in § 135 and the plane in which w is similarly represented. If $P(x, y)$ (Fig. 113) is a point on the z-plane and $P'(u, v)$ the corresponding point on the w-plane, the relation between P and P' is given by the equations

$$\begin{aligned} u &= u(x, y), \\ v &= v(x, y). \end{aligned} \tag{2}$$

Let $Q(x + dx, y + dy)$ be a point near P. Then $Q'(u + du, v + dv)$ is the corresponding point near P', where

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy,$$

and

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy. \tag{3}$$

Let $PQ = ds = \sqrt{dx^2 + dy^2}$

and $P'Q' = d\sigma = \sqrt{du^2 + dv^2}.$

Then, from (3) and the relations (3), § 142, it is easy to calculate that

$$d\sigma = M\,ds, \tag{4}$$

where

$$M = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} = \sqrt{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}. \tag{5}$$

Since the coefficient M depends only on the coördinates of P and not on those of Q, formula (4) shows that all infinitesimal lengths emanating from P are magnified in the same ratio. The scale of magnification changes, however, as the point P changes.

Let $R(x + \delta x,\ y + \delta y)$ be another point near P, and let $R'(u + \delta u,\ v + \delta v)$ be the corresponding point near P'. It is easy to show, by virtue of the relations (3), § 142, that

$$dx\,\delta x + dy\,\delta y = M(du\,\delta u + dv\,\delta v). \tag{6}$$

Consequently, if θ is the angle between PQ and PR, and θ' is the angle between $P'Q'$ and $P'R'$, we have, by (4), § 45,

$$\cos\theta = \cos\theta'.$$

Hence if two curves on the z-plane intersect at an angle θ, the corresponding curves on the w-plane intersect at the same angle. In other words, angles are preserved. For this reason the relation between the two planes is said to be *conformal.*

The discussion given above fails for points for which $M = 0$ or ∞. For such points we do not expect to find preservation of angle. By (4), § 142, M is the absolute value of $f'(z)$. Hence the conformal property fails at the points for which $f'(z) = 0$ or ∞.

Example 1. $w = z^2$.

Here
$$u = x^2 - y^2, \quad v = 2\,xy, \tag{7}$$
and
$$d\sigma = 2\sqrt{x^2 + y^2}\, ds. \tag{8}$$

The magnification is consequently symmetric about the origin and becomes greater as the point (x, y) is taken farther from the origin.

The representation will be conformal at all finite points except where $\dfrac{dw}{dz} = 2\,z = 0$. In fact, if we write
$$z = r(\cos\theta + i \sin\theta),$$
we have
$$w = r^2(\cos 2\,\theta + i \sin 2\,\theta);$$
whence it appears that the angle of w is twice the angle of z, so that if z describes an arc subtending an angle θ at the origin, w describes an arc which subtends an angle $2\,\theta$ at its origin. Hence the first quadrant of the z-plane is imaged on the upper half of the w-plane. In fact, from (7) it appears that if $y = 0$ and x varies from 0 to $+\infty$, then $v = 0$ and u varies from 0 to $+\infty$; if $x = 0$ and y varies from 0 to $+\infty$, then $v = 0$ and u varies from 0 to $-\infty$. Hence the positive part of the x-axis corresponds to the positive part of the u-axis, and the positive part of the y-axis corresponds to the negative part of the u-axis.

The straight lines
$$x = c_1, \quad y = c_2 \tag{9}$$
on the z-plane correspond to the two orthogonal families of parabolas
$$v^2 = -4\,c_1{}^2(u - c_1{}^2), \quad v^2 = 4\,c_2{}^2(u + c_2{}^2) \tag{10}$$
on the w-plane.

On the other hand, the straight lines
$$u = c_1, \quad v = c_2 \tag{11}$$
on the w-plane correspond to the two orthogonal families of hyperbolas
$$x^2 - y^2 = c_1, \quad xy = \tfrac{1}{2}\,c_2 \tag{12}$$
on the z-plane.

Example 2. $w = e^{iz}$.

Here
$$u = e^{-y}\cos x,$$
and
$$v = e^{-y}\sin x. \tag{13}$$

Since
$$e^{iz} = e^{iz + 2\pi i} = e^{i(z + 2\pi)},$$
all values of w are obtained by considering a strip of width $2\,\pi$ measured parallel to OX on the z-plane, the sides of the strip being parallel to OY. In other words, the entire w-plane is imaged on such a strip. The conformal property fails only when $z = \infty$. For other points
$$d\sigma = e^{-y} ds, \tag{14}$$
so that the magnification depends on the distance of z from the x-axis.

The straight lines $y = c_1$

parallel to the axis of reals on the z-plane correspond to circles

$$u^2 + v^2 = e^{-2c_1} \tag{15}$$

with center at the origin. If $c_1 = 0$ the radius of the circle (15) is unity, if $c_1 > 0$ the radius is less than unity, and if $c_1 < 0$ the radius is greater than unity. Hence we infer that the upper half of the z-plane corresponds to the portion of the w-plane inside a circle with radius unity, and the lower half of the z-plane corresponds to the portion of the w-plane outside the same circle. When $y = \infty$ the circle (15) becomes simply the origin on the w-plane, and when $y = -\infty$ the circle (15) has an infinite radius.

The straight lines $x = c_2$

parallel to the axis of imaginaries on the z-plane correspond to the straight lines

$$v = u \tan c_2 \tag{16}$$

on the w-plane which are orthogonal to the circles (15). To increase c_2 by 2π does not change the line (16), in agreement with the fact, already noted, that the w-plane corresponds to a strip of width 2π on the z-plane.

The relation between the z-plane and the w-plane is essentially that which exists between two maps of the earth's surface, one a stereographic projection and the other a Mercator projection. In the former, which may be taken as the w-plane, the north pole is the origin, the circles of latitude are concentric circles around the pole, and the meridian lines are straight lines through the pole. In Mercator's projection circles of latitude and longitude are straight lines, the north pole is at infinity, and the magnification, or distortion, becomes greater the closer one comes to the pole. The student is advised to compare two such maps, to be found in any atlas.

145. Integral of a complex function. Let $f(z)$ be analytic in a region R (Fig. 114) and consider the integral

$$\int_{z_1}^{z_2} f(z)dz \tag{1}$$

taken along a curve C drawn from z_1 to z_2. This is essentially a line integral. In fact, if we place

$$f(z) = u(x, y) + iv(x, y), \quad dz = dx + i\,dy,$$

(1) becomes

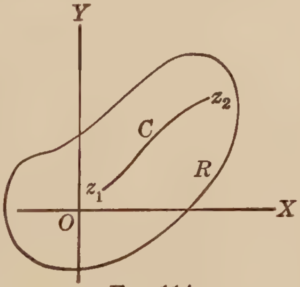

FIG. 114

$$\int_{(x_1, y_1)}^{(x_2, y_2)} (u\,dx - v\,dy) + i\int_{(x_1, y_1)}^{(x_2, y_2)} (v\,dx + u\,dy), \tag{2}$$

and the conditions that the integrals in (2) should be independent of the curve C are exactly the conditions that $f(z)$ should be an analytic function. We have, therefore, the following theorem:

In any region R in which $f(z)$ *is a single-valued continuous function of* z *the integral* $\int f(z)dz$ *is independent of the path of integration between* z_1 *and* z_2, *and the integral* $\int f(z)dz$ *around a closed path is zero.*

A corollary is that the path of this integral, whether closed or between fixed limits, may be deformed without changing the value of the integral, provided that in the deformation no point is encountered at which $f(z)$ ceases to be analytic.

146. Cauchy's theorem. Let $f(z)$ be single-valued and analytic in a region including a point $z = a$ and bounded by a curve C (Fig. 115). Draw a small circle around a as a center. Then in the area bounded by C and this circle the function $\dfrac{f(z)}{z-a}$ is continuous and single-valued. Hence

$$\int_{(C)} \frac{f(z)}{z-a}\,dz = \int_{\odot} \frac{f(z)}{z-a}\,dz, \qquad (1)$$

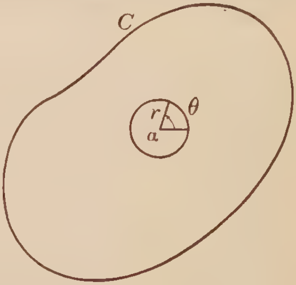

FIG. 115

where the second integral is taken around the small circle.

Now since $f(z)$ is continuous at $z = a$,

$$f(z) = f(a) + \epsilon.$$

Hence
$$\int_{\odot} \frac{f(z)}{z-a}\,dz = f(a)\int_{\odot} \frac{dz}{z-a} + \int_{\odot} \frac{\epsilon\,dz}{z-a}.$$

On the circumference of the circle we have

$$z - a = r(\cos\theta + i\sin\theta) = re^{i\theta},$$
$$dz = ire^{i\theta}\,d\theta,$$
$$\frac{dz}{z-a} = i\,d\theta.$$

Therefore

$$\int_{\odot} \frac{f(z)}{z-a}\,dz = f(a)\int_0^{2\pi} i\,d\theta + \int_0^{2\pi} i\epsilon\,d\theta = 2\pi i f(a) + \eta,$$

where $\eta = i\int_0^{2\pi} \epsilon\,d\theta$. Now we may take the radius of the circle so small that $|\epsilon|$ is less than any assigned value for all points on the circle. Hence $|\eta|$ is less than any assigned value, and the value of $\int_{\odot} \frac{f(z)}{(z-a)}\,dz$ differs from $2\,\pi i f(a)$ by a quantity which can be made as small as we please. Therefore, from (1),

$$f(a) = \frac{1}{2\,\pi i}\int_{(C)} \frac{f(z)}{z-a}\,dz. \tag{2}$$

This is *Cauchy's theorem.*

Another form of this result is

$$f(z) = \frac{1}{2\,\pi i}\int_{(C)} \frac{f(t)}{t-z}\,dt, \tag{3}$$

where z is held constant in the integration and t traverses the curve C.

It may be shown that the integral (3) may be differentiated under the integral sign and that each result thus obtained may be differentiated in the same way. We shall assume this. Then

$$\left.\begin{aligned}
f'(z) &= \frac{1}{2\,\pi i}\int_{(C)} \frac{f(t)}{(t-z)^2}\,dt,\\
f''(z) &= \frac{2!}{2\,\pi i}\int_{(C)} \frac{f(t)}{(t-z)^3}\,dt,\\
f'''(z) &= \frac{3!}{2\,\pi i}\int_{(C)} \frac{f(t)}{(t-z)^4}\,dt,\\
&\cdots\cdots\cdots\\
f^{(n)}(z) &= \frac{n!}{2\,\pi i}\int_{(C)} \frac{f(t)}{(t-z)^{n+1}}\,dt.
\end{aligned}\right\} \tag{4}$$

From these it follows that if a function is analytic, all its derivatives exist. This is not necessarily true for a function of a real variable.

147. Taylor's series. Let $f(z)$ be analytic within a circle C (Fig. 116) of center $z=a$ and radius R. By (3), § 146, if z is any point within C, we have

$$f(z) = \frac{1}{2\,\pi i}\int_{(C)} \frac{f(t)}{t-z}\,dt, \tag{1}$$

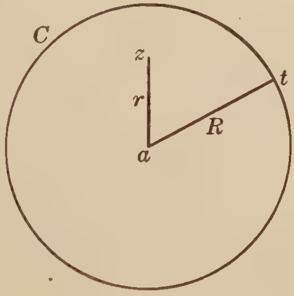

FIG. 116

where the integral is taken around the circle C. We have clearly

$$\frac{1}{t-z}=\frac{1}{t-a-(z-a)}=\frac{1}{t-a}\cdot\frac{1}{1-\dfrac{z-a}{t-a}}$$
$$=\frac{1}{t-a}\left[1+\frac{z-a}{t-a}+\frac{(z-a)^2}{(t-a)^2}+\cdots+\frac{(z-a)^n}{(t-a)^n}+\frac{\left(\dfrac{z-a}{t-a}\right)^{n+1}}{1-\dfrac{z-a}{t-a}}\right].$$

Substituting in (1), we have

$$f(z)=\frac{1}{2\pi i}\int_{(C)}\frac{f(t)}{t-a}\,dt+\frac{z-a}{2\pi i}\int_{(C)}\frac{f(t)}{(t-a)^2}\,dt+\cdots$$
$$+\frac{(z-a)^n}{2\pi i}\int_{(C)}\frac{f(t)dt}{(t-a)^{n+1}}+R_n, \tag{2}$$

where
$$R_n=\frac{1}{2\pi i}\int_{(C)}\frac{f(t)}{t-z}\left(\frac{z-a}{t-a}\right)^{n+1}dt. \tag{3}$$

By (3) and (4), §146, formula (2) is

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\cdots$$
$$+\frac{(z-a)^n}{n!}f^{(n)}(a)+R_n. \tag{4}$$

We now wish to show that

$$\lim_{n\to\infty} R_n=0. \tag{5}$$

Let $\qquad |z-a|=r \quad\text{and}\quad |t-a|=R.$

Then $\qquad |t-z|\geqq R-r.$

Also, let M be the largest value which $|f(t)|$ takes on the circumference of C. Then

$$\left|\frac{f(t)}{t-z}\left(\frac{z-a}{t-a}\right)^{n+1}\right|<\frac{M}{R-r}\left(\frac{r}{R}\right)^{n+1}. \tag{6}$$

Since $r<R$, we may, by taking n sufficiently great, make the expression (6) less than any assigned positive quantity ϵ. Hence, from (3),

$$|R_n|<\frac{\epsilon}{2\pi i}\int_{(C)}dt.$$

By using polar coördinates with center at a, $t-a=Re^{i\theta}$ and $dt=Rie^{i\theta}\,d\theta$. Hence

$$|R_n|<\frac{R\epsilon}{2\pi}\int_0^{2\pi}d\theta=R\epsilon,$$

from which (5) follows.

Hence (2) gives us the infinite Taylor series

$$f(z) = f(a) + (z - a)f'(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots, \tag{7}$$

which converges for all points within the circle C. The size of C is limited only by the condition that $f(z)$ shall be analytic within it. Hence the circle C may be extended until it meets the nearest singular point of $f(z)$. In this way the *circle of convergence* of (7) is determined. When a function may be expanded around $z = a$ in the series (7), it is said to be *regular* at the point a.

148. Poles and residues. An analytic function $f(z)$ is said to have a pole of order m at the point a if

$$f(z) = \frac{\phi(z)}{(z-a)^m}, \tag{1}$$

where $\phi(z)$ is a function which is regular at a. By expanding $\phi(z)$ into a Taylor series in the neighborhood of a we get from (1) a series of the form

$$f(z) = \frac{b_m}{(z-a)^m} + \cdots + \frac{b_1}{z-a} + a_0 + a_1(z-a) + \cdots, \tag{2}$$

or, since the series which closes (2) defines an analytic function $\psi(z)$,

$$f(z) = \frac{b_m}{(z-a)^m} + \frac{b_{m-1}}{(z-a)^{m-1}} + \cdots + \frac{b_1}{z-a} + \psi(z). \tag{3}$$

Consider now the integral

$$\int_{(C)} f(z)dz$$

taken along a closed path within which $f(z)$ is analytic except for the pole a. Then, by § 145,

$$\int_{(C)} \psi(z)dz = 0,$$

and, except in the case $m = -1$,

$$\int_{(C)} \frac{b_m}{(z-a)^m}\,dz = \left[\frac{b_m}{(1-m)(z-a)^{m-1}}\right]_z^z = 0,$$

since z returns to its original value by a complete circuit of C.

We have, therefore, $\displaystyle\int_{(C)} f(z)dz = \int_{(C)} \frac{b_1}{z-a}\,dz.$

To evaluate the last integral we may deform C into a circle (§ 115) with center at a and radius r (Fig. 117) and write

$$z - a = re^{i\theta},$$

$$\int_{(C)} f(z)dz = \int_{(C)} \frac{b_1}{z-a}\, dz = \int_0^{2\pi} b_1 i\, d\theta = 2\,\pi i b_1.$$

The quantity b_1, which is the only coefficient in the expansion (2) which affects the value of the integral of $f(z)$ around C, is called the *residue* of the pole.

Consider now a curve C (Fig. 118) surrounding any number of poles $a_1, a_2, a_3, \cdots, a_n$ of $f(z)$ and let $R_1, R_2, R_3, \cdots, R_n$ be the residues of the poles. The path of integration of $f(z)dz$ around C may be deformed into circles around $a_1, a_2, \cdots, a_n$, and, applying the result just obtained, we have

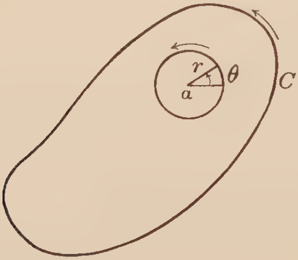

FIG. 117

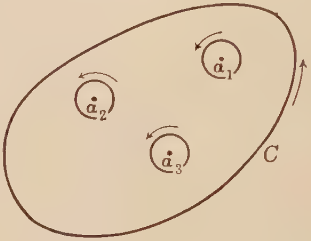

FIG. 118

$$\int_{(C)} f(z)dz = 2\,\pi i(R_1 + R_2 + \cdots + R_n).$$

That is, *the integral of an analytic function around a closed path in which the function has no singularities except poles is equal to* 2 *πi times the sum of the residues of the poles.*

The student should not think that a pole is the only singularity which a function may have. It is the only kind which we wish to consider. For a more complete study of singularities the student is referred to treatises on the theory of functions of a complex variable.

149. Application to real integrals. The theorem on residues (§ 148) may be used to evaluate certain integrals of real variables. We will show this by examples.

Example 1. Consider $\displaystyle\int \frac{e^{imz}}{1+z^2}\,dz,$

taken along the closed path (Fig. 119) formed by the axis of reals from $-R$ to R and a semicircle from R back to $-R$. Within this path the function integrated has a pole $z=i$. To find its residue we write

$$\frac{e^{imz}}{1+z^2}=\frac{1}{z-i}\left(\frac{e^{imz}}{z+i}\right)=\frac{1}{z-i}\,\phi(z)$$
$$=\frac{1}{z-i}\,[\phi(i)+(z-i)\phi'(i)+\cdots],$$

and the residue is

$$\phi(i)=\frac{e^{-m}}{2\,i}.$$

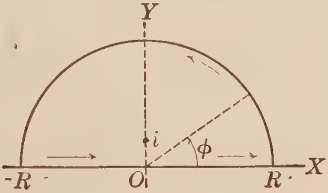

FIG. 119

Hence by the theorem of the last section the value of the integral along the path mentioned is πe^{-m}.

Along the axis of reals $z=x$, and along the semicircle

$$z=R(\cos\phi+i\sin\phi);$$

whence $\qquad e^{imz}=e^{-Rm\sin\phi}(\cos Rm\phi+i\sin Rm\phi).$

Consequently for the given integral along the closed path we have

$$\int_{-R}^{R}\frac{e^{imx}dx}{1+x^2}+\int_0^{\pi}\frac{e^{-Rm\sin\phi}(\cos Rm\phi+i\sin Rm\phi)}{1+R^2(\cos 2\,\phi+i\sin 2\,\phi)}\,R(-\sin\phi+i\cos\phi)d\phi$$
$$=\pi e^{-m}.$$

Now let $R\rightarrow\infty$. It is not difficult to see that the last integral approaches zero as a limit, and therefore

$$\int_{-\infty}^{\infty}\frac{e^{imx}}{1+x^2}\,dx=\pi e^{-m}.$$

But this is $\qquad \displaystyle\int_{-\infty}^{\infty}\left(\frac{\cos mx}{1+x^2}+i\,\frac{\sin mx}{1+x^2}\right)dx=\pi e^{-m},$

and equating real and imaginary parts, we have

$$\int_{-\infty}^{\infty}\frac{\cos mx}{1+x^2}\,dx=\pi e^{-m},$$

$$\int_{-\infty}^{\infty}\frac{\sin mx}{1+x^2}\,dx=0;$$

whence, finally, $\qquad \displaystyle\int_0^{\infty}\frac{\cos mx}{1+x^2}\,dx=\frac{\pi}{2}\,e^{-m}. \qquad (1)$

Example 2. Consider the integral

$$\int \frac{e^{iz}\,dz}{z}$$

along the path (Fig. 120) consisting of the axis of reals from r to $+R$, a semicircle from R to $-R$, the axis of reals from $-R$ to $-r$, and a semicircle from $-r$ to $+r$.

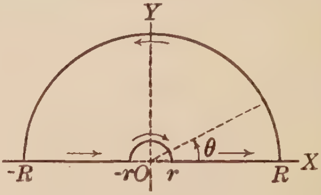

FIG. 120

Along the axis of reals $z = x$, and along the two semicircles

$$z = R(\cos\theta + i\sin\theta) = Re^{i\theta},$$

and $\quad z = r(\cos\theta + i\sin\theta) = re^{i\theta},$

respectively.

Since the function has no pole in the region bounded by the path, the integral is zero, and we have

$$\int_r^R \frac{e^{ix}}{x}\,dx + \int_0^\pi e^{-R\sin\theta}(\cos R\theta + i\sin R\theta)i\,d\theta$$
$$+ \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_\pi^0 e^{-r\sin\theta}(\cos r\theta + i\sin r\theta)i\,d\theta = 0.$$

Now let $R \rightarrow \infty$, $r \rightarrow 0$. It is easy to see that the second integral approaches 0 and that the fourth integral approaches $-\pi i$. Hence

$$\int_0^\infty \frac{e^{ix}}{x}\,dx + \int_{-\infty}^0 \frac{e^{ix}}{x}\,dx - \pi i = 0. \tag{2}$$

In the second integral let $x = -\lambda$. Then

$$\int_\infty^0 \frac{e^{-i\lambda}}{-\lambda}\,d(-\lambda) = -\int_0^\infty \frac{e^{-i\lambda}}{\lambda}\,d\lambda = -\int_0^\infty \frac{e^{-ix}}{x}\,dx,$$

so that (2) is
$$\int_0^\infty \frac{e^{ix} - e^{-ix}}{x}\,dx = \pi i;$$

that is
$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \tag{3}$$

Example 3. Consider the integral

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx,$$

which we have used in the Gamma functions. This converges if p is positive and less than unity. To evaluate it we take the integral

$$\int \frac{z^{p-1}}{1+z}\,dz$$

along a closed path (Fig. 121) consisting of (1) the axis of reals from r to R, (2) a circle from R back to R, (3) the axis of reals from R to r, and (4) a circle from r to r.

Inside this boundary the function has one pole, $z = -1$, and the residue is

$$(-1)^{p-1} = e^{(p-1)\pi i} = \cos (p-1)\pi + i \sin (p-1)\pi,$$

so that the value of the integral is

$$-2\pi \sin (p-1)\pi + 2\pi i \cos (p-1)\pi.$$

Consider each of the four paths in succession.

Along path (1) $z = x = x(\cos 0 + i \sin 0)$. Therefore the integral is

$$\int_r^R \frac{x^{p-1}}{1+x}\,dx.$$

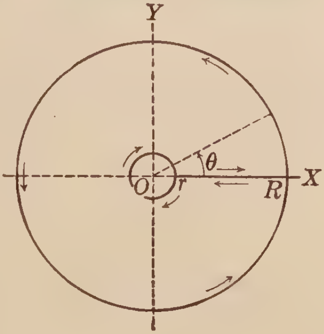

FIG. 121

Along path (2) $z = Re^{i\theta}$. The integral is

$$\int_0^{2\pi} \frac{R^{p-1}e^{i(p-1)\theta}}{1+Re^{i\theta}}\,d\theta,$$

and the limit of this is zero as $R \to \infty$, since $p < 1$.

Along path (3) $z = x$; but we must now write

$$z = x(\cos 2\pi + i \sin 2\pi) = xe^{2\pi i},$$

since the angle of z has been increased by 2π by the passage around the circle R. Hence the integral is

$$\int_R^r \frac{x^{p-1}e^{2(p-1)\pi i}}{1+x}\,dx = \int_R^r \frac{x^{p-1}[\cos 2(p-1)\pi + i \sin 2(p-1)\pi]}{1+x}\,dx.$$

Along path (4), $x = re^{i\theta}$. The integral is

$$\int_{2\pi}^0 \frac{r^{p-1}e^{i(p-1)\theta}}{1+re^{i\theta}}\,ri\,d\theta,$$

and the limit of this is zero as $r \to 0$, since $p > 0$.

Putting together the four results and at the same time passing to the limit, we have

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx + \int_\infty^0 \frac{x^{p-1}\cos 2(p-1)\pi}{1+x}\,dx + i\int_\infty^0 \frac{x^{p-1}\sin 2(p-1)\pi}{1+x}\,dx$$
$$= -2\pi \sin (p-1)\pi + 2\pi i \cos (p-1)\pi.$$

Equating real parts and making simple reductions, we have

$$(1 - \cos 2p\pi)\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = 2\pi \sin p\pi,$$

or

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{2\pi \sin p\pi}{2\sin^2 p\pi} = \frac{\pi}{\sin p\pi}. \tag{4}$$

150. Application to Bessel functions. We have seen in (4), §116, that

$$J_n(x) = \frac{x^n}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_{-1}^{1} e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt \tag{1}$$

is a solution of the Bessel equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0. \tag{2}$$

This leads us to inquire whether other integrals of the form

$$y = x^n\int_{\alpha}^{\beta} e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt \tag{3}$$

may be solutions of the same equation if α and β are properly determined. We will accordingly substitute (3) in (2). We obtain

$$(2n+1)x^{n+1}\int_{\alpha}^{\beta} ite^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt + x^{n+2}\int_{\alpha}^{\beta} e^{ixt}(1-t^2)^{n+\frac{1}{2}}\,dt = 0. \tag{4}$$

The first integral in (4) may be integrated by parts, using $u = ie^{ixt}$ and $dv = t(1-t^2)^{n-\frac{1}{2}}\,dt$. Then (4) reduces to

$$\Big[-ix^{n+1}(1-t^2)^{n-\frac{1}{2}}e^{ixt}\Big]_{t=\alpha}^{t=\beta} = 0. \tag{5}$$

Equation (5) may be satisfied by placing $\alpha = -1$, $\beta = 1$. In that case we have the integral which occurs in the function $J_n(x)$. Equation (5) may also be satisfied by placing $\alpha = 1$, $\beta = 1 + i\infty$, and we have a solution of (2) in the form

$$y = x^n\int_{1}^{1+i\infty} e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt. \tag{6}$$

To study this we note that the line integral of

$$\int e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt$$

is zero if taken around the closed path $OPQRSO$ (Fig. 122) in the plane $t = \xi + i\eta$. We leave it to the student to show that the integral around the quarter-circle PQ approaches zero as a limit as the radius of the circle approaches zero, and that the

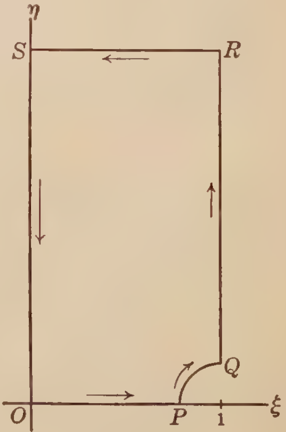

FIG. 122

integral along RS approaches zero as OS approaches infinity. Hence we have

$$\int_1^{1+i\infty} e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt = -\int_\infty^0 e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt - \int_0^1 e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt, \tag{7}$$

where the first integral is taken along the line QR, the second along the line SO, and the third along the line OP.

If we take x as real, the first integral on the right of (7) is a pure imaginary, since it is taken along the axis of imaginaries where e^{ixt} and $(1-t^2)^{n-\frac{1}{2}}$ are real and dt is a pure imaginary.

The second integral on the right of (7) breaks up into a real integral

$$\int_0^1 \cos xt(1-t^2)^{n-\frac{1}{2}}\,dt$$

and a pure imaginary integral

$$i\int_0^1 \sin xt(1-t^2)^{n-\frac{1}{2}}\,dt.$$

Hence we have, using $\mathcal{R}(y)$ to denote the real part of y,

$$\begin{aligned}\mathcal{R}(y) &= -x^n\int_0^1 \cos xt(1-t^2)^{n-\frac{1}{2}}\,dt\\ &= -\frac{x^n}{2}\int_{-1}^1 \cos xt(1-t^2)^{n-\frac{1}{2}}\,dt\\ &= -\frac{x^n}{2}\int_{-1}^1 e^{ixt}(1-t^2)^{n-\frac{1}{2}}\,dt,\end{aligned}$$

the last transformation being made as was done in obtaining (11), § 114, and therefore, by (1),

$$J_n(x) = -\frac{1}{2^{n-1}\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\mathcal{R}(y). \tag{8}$$

To obtain the real part of y substitute in (6)

$$t = 1 + \frac{iv}{x},$$

where v is a new variable. We have, in the first place,

$$y = -x^{n-1}\int_0^\infty (-i)e^{ix-v}\left(-\frac{2\,iv}{x}+\frac{v^2}{x^2}\right)^{n-\frac{1}{2}}dv.$$

If we take out of the parenthesis the factor $-\frac{2\,iv}{x}$ and place $-i=e^{-\frac{\pi}{2}i}$, then after a few changes we have

$$y=-\frac{e^{i\psi}2^{n-\frac{1}{2}}}{x^{\frac{1}{2}}}\int_0^\infty e^{-v}v^{n-\frac{1}{2}}\left(1+\frac{iv}{2\,x}\right)^{n-\frac{1}{2}}dv, \tag{9}$$

where $$\psi=x-\frac{(2\,n+1)\pi}{4}.$$

Let the quantity in parenthesis be expanded by the binomial series. The typical term is

$$\frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(n-\frac{2\,k-1}{2}\right)}{k!\,(2\,x)^k}(iv)^k, \tag{10}$$

which must be taken equal to unity when $k=0$. Therefore

$$\begin{aligned}y&=-\frac{e^{i\psi}2^{n-\frac{1}{2}}}{x^{\frac{1}{2}}}\sum_{k=0}^{k=\infty}\left[\frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(n-\frac{2k-1}{2}\right)i^k}{k!\,(2\,x)^k}\int_0^\infty e^{-v}v^{n+k-\frac{1}{2}}\,dv\right]\\
&=-\frac{e^{i\psi}2^{n-\frac{1}{2}}}{x^{\frac{1}{2}}}\sum_{k=0}^{k=\infty}\left[\frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(n-\frac{2\,k-1}{2}\right)i^k}{k!\,(2\,x)^k}\Gamma\left(n+k+\frac{1}{2}\right)\right]\\
&=-\frac{e^{i\psi}2^{n-\frac{1}{2}}}{x^{\frac{1}{2}}}\sum_{k=0}^{k=\infty}i^k\left[\frac{\left(n^2-\frac{1}{4}\right)\left(n^2-\frac{9}{4}\right)\cdots\left(n^2-\frac{(2\,k-1)^2}{4}\right)}{k!\,(2\,x)^k}\Gamma\left(n+\frac{1}{2}\right)\right]\\
&=-\frac{e^{i\psi}2^{n-\frac{1}{2}}}{x^{\frac{1}{2}}}[P(x)+iQ(x)]\Gamma\left(n+\frac{1}{2}\right),\end{aligned}$$

where

$$P(x)=1-\frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})}{2!\,(2\,x)^2}+\frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})(n^2-\frac{25}{4})(n^2-\frac{49}{4})}{4!\,(2\,x)^4}-\cdots,$$

$$Q(x)=\frac{n^2-\frac{1}{4}}{2\,x}-\frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})(n^2-\frac{25}{4})}{3!\,(2\,x)^3}+\cdots.$$

Therefore the real part of y is

$$(-P\cos\psi+Q\sin\psi)\,\frac{2^{n-\frac{1}{2}}}{x^{\frac{1}{2}}}\,\Gamma\left(n+\frac{1}{2}\right),$$

and, from (8), $$J_n(x)=\sqrt{\frac{2}{\pi x}}\,(P\cos\psi-Q\sin\psi). \tag{11}$$

The solution (11) is one which may be used to compute $J_n(x)$ for large values of x. The series for P and Q do not, however, converge; but it may be shown that the error made in neglecting the terms after a sufficient number is less than the first term neglected, so that if the series is broken off before its smallest term the best degree of accuracy is obtained.

To prove the statement just made let R_k be the remainder in the expansion of $\left(1+\frac{iv}{2x}\right)^{n-\frac{1}{2}}$ after the term (10) is reached. Then, by (12), § 7,

$$R_k=\frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(n-\frac{2k+1}{2}\right)}{k!}\left(\frac{i}{2x}\right)^{k+1}\int_0^v t^k\left(1+\frac{i(v-t)}{2x}\right)^{n-k-\frac{3}{2}}dt.$$

But $\left(1+\frac{i(v-t)}{2x}\right)^{n-k-\frac{3}{2}}<1$ in absolute value if $k>n+\frac{3}{2}$ and t lies between 0 and v. Hence

$$|R_k|<\frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(n-\frac{2k+1}{2}\right)}{(k+1)!(2x)^{k+1}}v^{k+1}$$

and

$$\left|\int_0^\infty e^{-v}v^{n-\frac{1}{2}}R_k\,dv\right|$$
$$<\frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(n-\frac{2k+1}{2}\right)}{(k+1)!(2x)^{k+1}}\Gamma\left(n+k+\frac{1}{2}\right);$$

that is,

$$\left|\int_0^\infty e^{-v}v^{n-\frac{1}{2}}R_k\,dv\right|$$
$$<\frac{\left(n^2-\frac{1}{4}\right)\left(n^2-\frac{9}{4}\right)\cdots\left(n^2-\frac{(2k+1)^2}{4}\right)}{(k+1)!(2x)^{k+1}}\Gamma\left(n+\frac{1}{2}\right).$$

From this it follows that the error made in cutting off the series

$$P+iQ$$

with any given term is less in absolute magnitude than the value of the first term omitted. The expansion for $J_n(x)$ has, then, the same property. Such a series is an example of an *asymptotic expansion.*

We may find another solution of the Bessel equation by taking the imaginary part of the integral y multiplied by any constant. In that way we find the solution

$$Y_n(x)=\sqrt{\frac{2}{\pi x}}\,[Q(x)\cos\psi+P(x)\sin\psi].$$

EXERCISES

Carry out the following operations and graph each of the given numbers and the results:

1. $(2+3i)+(4-i)$. 2. $(2+3i)-(4-i)$. 3. $\dfrac{1+i}{1-i}$. 4. $\dfrac{2+3i}{i}$.

5. Find the modulus and the angle of each of the following numbers: $1+\sqrt{-3},\ 1-\sqrt{-3},\ -4,\ -7\sqrt{-1}$.

Find the following powers and express the results graphically:

6. $(2-3i)^2$. 7. $(1-i\sqrt{2})^2$. 8. $(1+i)^3$. 9. $(1-i)^4$.

Find all the values of the following indicated roots and locate them graphically:

10. $\sqrt[4]{1}$. 11. $\sqrt[4]{-1}$. 12. $\sqrt[5]{32}$. 13. $\sqrt[5]{-32}$. 14. $\sqrt[3]{-8}$. 15. $\sqrt[3]{8}$.

16. If $1, \omega_1, \omega_2$ are the three cube roots of unity, prove that $\omega_2{}^2=\omega_1$, $\omega_1{}^2=\omega_2$, $1+\omega_1+\omega_2=0$.

17. Study the effect on $w=\sqrt[3]{z}$ by various paths described by z.

Express the following as complex numbers:

18. $e^{\frac{\pi}{4}i}$. 20. $\sin(1+i)$. 22. $\sinh i$. 24. $\log(-2)$.

19. e^{i-1}. 21. $\cos\left(-\frac{1}{2}+\frac{1}{2}\sqrt{-3}\right)$. 23. $\cosh\left(1+\sqrt{2}\right)$. 25. $\log(1-i)$.

Find the orthogonal systems of curves defined by the real and imaginary parts of the following functions:

26. $\dfrac{1}{z}$. 27. $\log\dfrac{z-1}{z+1}$. 28. $\log\sqrt{z^2-1}$. 29. $\sqrt{z}$.

Study the conformal mapping defined by the following functions:

30. $w=z^n$. 32. $w=\sin z$. 34. $w=\dfrac{z-1}{z+1}$.

31. $w=\log z$. 33. $w=\dfrac{1}{z}$. 35. $w=\cosh z$.

Calculate the following integrals, where m, a, and b are real numbers:

36. $\displaystyle\int_0^\infty \frac{\sin mx}{x(x^2+a^2)^2}\,dx$. 39. $\displaystyle\int_{-\infty}^\infty \frac{e^{ax}-e^{bx}}{1-e^x}\,dx$. $(a<1,\ b<1)$

37. $\displaystyle\int_{-\infty}^\infty \frac{\cos mx}{1+x^4}\,dx$. 40. $\displaystyle\int_0^\infty \frac{\cos x}{a^2+x^2}$. $(a>0)$

38. $\displaystyle\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}$. $(a<1)$ 41. $\displaystyle\int_0^\infty \cos x^2\,dx=\int_0^\infty \sin x^2\,dx$.

CHAPTER XVI

ELLIPTIC INTEGRALS

151. Introduction. Any integral of the type

$$\int P(x)dx, \tag{1}$$

where $P(x)$ is an algebraic polynomial, is easily evaluated, and only one type of integral occurs; namely,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}. \tag{2}$$

Any integral of the type

$$\int \frac{P(x)}{Q(x)}\,dx, \tag{3}$$

where $P(x)$ and $Q(x)$ are polynomials, may be evaluated by separation into a polynomial and partial fractions. There is necessary a new type of integral, namely,

$$\int \frac{dx}{x} = \log x, \tag{4}$$

so that the integration of a rational fraction is only possible by aid of a new kind of function, the logarithm. In elementary work there also arises the type

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, \tag{5}$$

but from the standpoint of the complex variable this is not essentially different from (4).

Any integral of the type

$$\int R(x, \sqrt[n]{ax+b})dx, \tag{6}$$

where $R(x, \sqrt[n]{ax+b})$ is a rational function of x and $\sqrt[n]{ax+b}$, is integrable. For if we place

$$z = \sqrt[n]{ax+b},$$

we reduce (6) to the type (1) or (3).

Any integral of the type

$$\int R(x, \sqrt{ax^2 + bx + c}) \tag{7}$$

is integrable. For we may write

$$ax^2 + bx + c = a(x - r_1)(x - r_2)$$

and place

$$z(x - r_1) = \sqrt{a(x - r_1)(x - r_2)}.$$

Then

$$x = \frac{r_1 z^2 - a r_2}{z^2 - a},$$

$$\sqrt{ax^2 + bx + c} = \frac{a(r_1 - r_2)z}{z^2 - a},$$

$$dx = \frac{2\,a(r_2 - r_1)z}{(z^2 - a)^2}\,dz,$$

and (7) is reduced to the type (1) or (3). This proves the possibility of the integration, but does not outline the method which is necessarily the most convenient in practice. No essentially new type of integrals or functions arises, but it is convenient in elementary work to have the formulas

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}, \quad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\,(x + \sqrt{x^2 \pm a^2}). \tag{8}$$

This is as far as we can go in general statements as to the integrability of algebraic functions. If the integrand involves the nth root ($n > 2$) of a polynomial higher than the first degree, or the square root of a polynomial higher than the second degree, the integral cannot in general be evaluated in terms of elementary functions. Of course, particular cases of such integrals may sometimes be evaluated.

The integrals

$$\int R(x, \sqrt{ax^3 + bx^2 + cx + e})dx \tag{9}$$

and

$$\int R(x, \sqrt{ax^4 + bx^3 + cx^2 + ex + f})dx \tag{10}$$

are called *elliptic integrals*, and their evaluation requires new functions, the *elliptic functions*.

It may be shown that (9) may be reduced to (10) by algebraic substitutions, and that the integration of (10) may be reduced to the evaluation of integrals of elementary types and the following new types:

1. Elliptic integral of the first kind:

$$\int \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}. \tag{11}$$

2. Elliptic integral of the second kind:

$$\int\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx. \tag{12}$$

3. Elliptic integral of the third kind:

$$\int\frac{dx}{(x^2-a)\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{13}$$

These are Legendre's normal forms. In all of these it is usual to take $k < 1$.

We shall show later that (11) may be reduced to

$$\int\frac{dy}{\sqrt{4\,y^3-g_2y-g_3}}, \tag{14}$$

which is Weierstrass's normal form for an elliptic integral of the first kind.

152. The functions sn *u*, cn *u*, dn *u*. Consider the elliptic integral of the first kind:

$$u=\int_0^x\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{1}$$

This integral defines u as a function of k and x:

$$u = F(k, x).$$

The quantity k is the *modulus* of the integral. We shall consider it fixed and consider u as a function of x only. Conversely, x is a function of u defined by the integral (1). We use the symbol sn u for this function and have, from (1),

$$x = \operatorname{sn} u. \tag{2}$$

Involved in (1) are also the expressions $\sqrt{1-x^2}$ and $\sqrt{1-k^2x^2}$, giving other elliptic functions

$$\sqrt{1-x^2}=\sqrt{1-\operatorname{sn}^2 u}=\operatorname{cn} u \tag{3}$$

and

$$\sqrt{1-k^2x^2}=\sqrt{1-k^2\operatorname{sn}^2 u}=\operatorname{dn} u. \tag{4}$$

There are questions of algebraic signs to be given to the radicals involved in (3) and (4) which are partially answered by the statements taken as part of the definitions,

$$\begin{aligned}\operatorname{sn} 0 &= 0,\\ \operatorname{cn} 0 &= 1,\\ \operatorname{dn} 0 &= 1.\end{aligned} \tag{5}$$

All other values come out of these by continuous variation of x, as in § 137.

In (1) we may place $x = \sin\phi$.

We have
$$u = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}. \tag{6}$$

This defines u as a function of ϕ and, conversely, ϕ as a function of u, which is called the *amplitude* of u and is written

$$\phi = \text{am } u.$$

Then
$$\begin{aligned} x &= \sin(\text{am } u) = \text{sn } u, \\ \sqrt{1 - x^2} &= \cos(\text{am } u) = \text{cn } u, \\ \sqrt{1 - k^2x^2} &= \sqrt{1 - k^2 \sin^2\phi} = \text{dn } u. \end{aligned} \tag{7}$$

Let us now consider the effect on u in (6) by adding π to ϕ.

Let
$$u_1 = \int_0^{\phi_1} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}. \tag{8}$$

Then
$$\begin{aligned} \int_0^{\phi_1 + \pi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} &= \int_0^{\pi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} \\ &\quad + \int_{\pi}^{\pi + \phi_1} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}. \end{aligned} \tag{9}$$

The first integral on the right-hand side of (9) is obviously equal to

$$2\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}.$$

In the last integral in (9) place $\phi = \pi + \psi$. It becomes

$$\int_0^{\phi_1} \frac{d\psi}{\sqrt{1 - k^2 \sin^2\psi}},$$

which is the original u_1. Hence if we place

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}, \tag{10}$$

we have, from (9),

$$\int_0^{\phi_1 + \pi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} = 2K + u_1;$$

whence
$$\phi_1 + \pi = \text{am } (u_1 + 2K),$$

and consequently
$$\begin{aligned} \text{sn } (u_1 + 2K) &= -\text{sn } u_1, \\ \text{cn } (u_1 + 2K) &= -\text{cn } u_1, \\ \text{dn } (u_1 + 2K) &= \text{dn } u_1, \end{aligned} \tag{11}$$

since, by (8),
$$\phi_1 = \text{am } u_1.$$

By repetition of these formulas we have

$$\begin{aligned} \operatorname{sn}(u+4K) &= \operatorname{sn} u, \\ \operatorname{cn}(u+4K) &= \operatorname{cn} u, \\ \operatorname{dn}(u+4K) &= \operatorname{dn} u. \end{aligned} \tag{12}$$

The elliptic functions sn u, cn u have therefore the period 4 K, and the function dn u has the period 2 K.

These are not the only periods, however, as will be seen later.

153. Application to the pendulum. Let a simple pendulum of length l swing in an arc of a circle.

Let A (Fig. 123) be the lowest point of the bob, B its highest point, and P its variable position. Let the angle $AOB = \alpha$, the angle $AOP = \theta$, and let $OA = OP = OB = l$.

The differential equation of the motion is

$$l^2 \frac{d^2\theta}{dt^2} = -g \sin\theta;$$

whence

$$l\left(\frac{d\theta}{dt}\right)^2 = 2g(\cos\theta - \cos\alpha), \quad (1)$$

the constant of integration being determined by the fact that when $\theta = \alpha$ the velocity is zero.

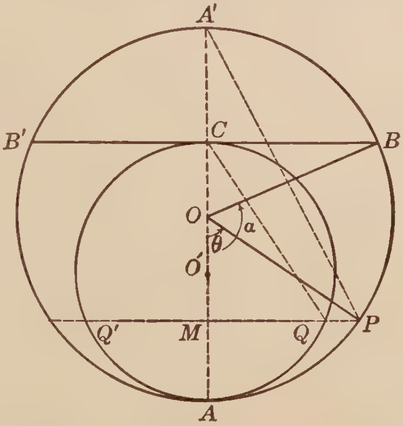

FIG. 123

From (1) we get
$$\sqrt{\frac{g}{l}}\, t = \int_0^{\theta} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}}, \tag{2}$$

if we assume that $\theta = 0$ when $t = 0$.

In (2) place
$$k = \sin\frac{\alpha}{2}, \qquad \frac{1}{k}\sin\frac{\theta}{2} = \sin\phi. \tag{3}$$

We have
$$\sqrt{\frac{g}{l}}\, t = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}. \tag{4}$$

Hence
$$\phi = \operatorname{am}\sqrt{\frac{g}{l}}\, t. \tag{5}$$

A geometric interpretation of many of the quantities involved in this problem may be given.

From B draw BB' perpendicular to OA and intersecting OA (produced if necessary) in C. On AC as a diameter describe a circle with center at O'. From P draw PM perpendicular to OA and intersecting the circle O' in Q. Draw CQ and prolong AO to meet the circle O at A', and draw $A'P$.

Now
$$\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{l-OM}{2l}}=\sqrt{\frac{AM}{2l}},$$
$$k=\sin\frac{\alpha}{2}=\sqrt{\frac{1-\cos\alpha}{2}}=\sqrt{\frac{l+OC}{2l}}=\sqrt{\frac{AC}{2l}}$$

(the last result is also true if C falls between A and O),
$$\sin\phi=\frac{1}{k}\sin\frac{\theta}{2}=\sqrt{\frac{AM}{AC}}=\sqrt{\frac{AM\cdot MC}{AC\cdot MC}}=\frac{MQ}{CQ}.$$

Hence
$$\phi=\text{angle } OCQ=\text{am}\sqrt{\frac{g}{l}}\,t. \tag{6}$$

Then
$$\text{sn}\sqrt{\frac{g}{l}}\,t=\sin\phi=\frac{MQ}{CQ}, \tag{7}$$
$$\text{cn}\sqrt{\frac{g}{l}}\,t=\cos\phi=\frac{CM}{CQ}, \tag{8}$$

and, with the aid of (3),
$$\text{dn}\sqrt{\frac{g}{l}}\,t=\sqrt{1-k^2\sin^2\phi}=\cos\frac{\theta}{2}=\frac{MA'}{A'P}, \tag{9}$$

since the angle $PA'A$ is $\frac{\theta}{2}$.

The construction is to be such that as P travels back and forth in its swing the point Q describes the smaller circle in a positive direction and the angle ϕ varies continuously from 0 to 2π.

As ϕ increases from 0 to $\frac{\pi}{2}$, the pendulum bob P swings from A to B, and t increases from 0 to $\sqrt{\frac{g}{l}}\,K$. As ϕ then increases from $\frac{\pi}{2}$ to π, P swings back to A, and t becomes $2\sqrt{\frac{g}{l}}\,K$. Then as ϕ increases to $\frac{3}{2}\pi$ and then to 2π, P swings up to B' and back to A, and t becomes successively $3\sqrt{\frac{g}{l}}\,K$ and $4\sqrt{\frac{g}{l}}\,K$.

Hence if we take as usual $4T$ as the period of the swing, we have
$$T=\sqrt{\frac{g}{l}}\,K.$$

From (11), § 152,

$$\begin{aligned} \operatorname{sn}\sqrt{\frac{g}{l}}\,(t+2\,T) &= -\operatorname{sn}\sqrt{\frac{g}{l}}\,t, \\ \operatorname{cn}\sqrt{\frac{g}{l}}\,(t+2\,T) &= -\operatorname{cn}\sqrt{\frac{g}{l}}\,t, \\ \operatorname{dn}\sqrt{\frac{g}{l}}\,(t+2\,T) &= \operatorname{dn}\sqrt{\frac{g}{l}}\,t. \end{aligned} \tag{10}$$

These results are directly evident from the figure; for if at the time t the point Q is as shown in the figure, it will be at Q' when the time is $t+T$. Then $MQ' = -MQ$, and CM is now minus the cosine of the corresponding angle ϕ. From (7), (8), (9) we may deduce (10) geometrically.

154. Formulas of differentiation and series expansion. From

$$u = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

we get
$$\frac{du}{dx} = \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}};$$

whence, by inverting,
$$\frac{d(\operatorname{sn} u)}{du} = \operatorname{cn} u \operatorname{dn} u. \tag{1}$$

From
$$\operatorname{cn} u = \sqrt{1-\operatorname{sn}^2 u}$$

we get
$$\frac{d(\operatorname{cn} u)}{du} = -\operatorname{sn} u \operatorname{dn} u, \tag{2}$$

and from
$$\operatorname{dn} u = \sqrt{1-k^2\operatorname{sn}^2 u}$$

$$\frac{d(\operatorname{dn} u)}{du} = -k^2\operatorname{sn} u \operatorname{cn} u. \tag{3}$$

From these we may get Maclaurin's series

$$\operatorname{sn} u = u - (1+k^2)\frac{u^3}{3!} + (1+14\,k^2+k^4)\frac{u^5}{5!} - \cdots, \tag{4}$$

$$\operatorname{cn} u = 1 - \frac{u^2}{2!} + (1+4\,k^2)\frac{u^4}{4!} - (1+44\,k^2+16\,k^4)\frac{u^6}{6!} + \cdots, \tag{5}$$

$$\operatorname{dn} u = 1 - k^2\frac{u^2}{2!} + k^2(4+k^2)\frac{u^4}{4!} - k^2(16+44\,k^2+k^4)\frac{u^6}{6!} + \cdots. \tag{6}$$

From these series follow the formulas, which may also be obtained from the original definitions,

$$\operatorname{sn}(-u) = -\operatorname{sn} u, \tag{7}$$

$$\operatorname{cn}(-u) = \operatorname{cn} u, \tag{8}$$

$$\operatorname{dn}(-u) = \operatorname{dn} u. \tag{9}$$

155. Addition formulas. Suppose u and v to vary so that

$$u + v = a,$$

where a is a constant.

Then
$$\frac{dv}{du} = -1.$$

Let
$$\begin{aligned} s_1 &= \operatorname{sn} u, & s_2 &= \operatorname{sn} v, \\ \dot{s}_1 &= \frac{ds_1}{du} & \dot{s}_2 &= \frac{ds_2}{du} = -\frac{ds_2}{dv} \\ &= \operatorname{cn} u \operatorname{dn} u. & &= -\operatorname{cn} v \operatorname{dn} v. \end{aligned}$$

Then
$$\begin{aligned} \dot{s}_1^2 &= (1 - s_1^2)(1 - k^2 s_1^2), & \dot{s}_2^2 &= (1 - s_2^2)(1 - k^2 s_2^2), \\ \ddot{s}_1 &= -(1 + k^2) s_1 + 2k^2 s_1^3, & \ddot{s}_2 &= -(1 + k^2) s_2 + 2k^2 s_2^3, \end{aligned}$$

where
$$\ddot{s}_1 = \frac{d\dot{s}_1}{du}, \qquad \ddot{s}_2 = \frac{d\dot{s}_2}{du}.$$

Then
$$\begin{aligned} \ddot{s}_1 s_2 - \ddot{s}_2 s_1 &= 2k^2 s_1 s_2 (s_1^2 - s_2^2), \\ \dot{s}_1^2 s_2^2 - \dot{s}_2^2 s_1^2 &= (1 - k^2 s_1^2 s_2^2)(s_2^2 - s_1^2), \end{aligned}$$

and
$$\frac{\ddot{s}_1 s_2 - \ddot{s}_2 s_1}{\dot{s}_1 s_2 - \dot{s}_2 s_1} = -\frac{2k^2 s_1 s_2 (\dot{s}_1 s_2 + \dot{s}_2 s_1)}{1 - k^2 s_1^2 s_2^2};$$

whence
$$\log(\dot{s}_1 s_2 - \dot{s}_2 s_1) = \log(1 - k^2 s_1^2 s_2^2) + C_1.$$

That is,
$$\frac{\dot{s}_1 s_2 - \dot{s}_2 s_1}{1 - k^2 s_1^2 s_2^2} = C,$$

or, written out in full,

$$\frac{\operatorname{cn} u \operatorname{dn} u \operatorname{sn} v + \operatorname{cn} v \operatorname{dn} v \operatorname{sn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} = C.$$

This is one solution of the differential equation

$$du + dv = 0,$$

of which another solution is evidently

$$u + v = a.$$

Hence, by the theory of differential equations, C must be a function of a; that is,

$$\frac{\operatorname{cn} u \operatorname{dn} u \operatorname{sn} v + \operatorname{cn} v \operatorname{dn} v \operatorname{sn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} = f(u + v).$$

To see what function this is we place $v = 0$ and find

$$\operatorname{sn} u = f(u);$$

therefore f is the function sn. Hence

$$\operatorname{sn}(u + v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \tag{1}$$

From this we find that

$$\operatorname{cn}(u+v)=\frac{\operatorname{cn} u \operatorname{cn} v-\operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{2}$$

$$\operatorname{dn}(u+v)=\frac{\operatorname{dn} u \operatorname{dn} v-k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \tag{3}$$

By the use of (7), (8), (9), § 154, the formulas for sn $(u-v)$, cn $(u-v)$, dn $(u-v)$ are easily written.

156. The periods. We have already defined K by the formula

$$K=\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{1}$$

From (1), $\operatorname{sn} K=1, \quad \operatorname{cn} K=0, \quad \operatorname{dn} K=k',$ (2)

where $k'=\sqrt{1-k^2}$, the real positive root being taken if $k<1$.

Using these values in the addition formulas, § 155, we have

$$\begin{aligned}\operatorname{sn}(u+K)&=\frac{\operatorname{cn} u}{\operatorname{dn} u},\\ \operatorname{cn}(u+K)&=-k'\frac{\operatorname{sn} u}{\operatorname{dn} u},\\ \operatorname{dn}(u+K)&=\frac{k'}{\operatorname{dn} u}.\end{aligned} \tag{3}$$

By adding K to u in (3) and again applying (3), we get

$$\begin{aligned}\operatorname{sn}(u+2K)&=-\operatorname{sn} u,\\ \operatorname{cn}(u+2K)&=-\operatorname{cn} u,\\ \operatorname{dn}(u+2K)&=\operatorname{dn} u;\end{aligned} \tag{4}$$

and again, adding $2K$,

$$\begin{aligned}\operatorname{sn}(u+4K)&=\operatorname{sn} u,\\ \operatorname{cn}(u+4K)&=\operatorname{cn} u,\\ \operatorname{dn}(u+4K)&=\operatorname{dn} u,\end{aligned} \tag{5}$$

in full accord with § 152.

We define K' by the formula

$$K'=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}. \tag{6}$$

Let us place
$$x = \frac{1}{\sqrt{1 - k'^2t^2}},$$
$$\sqrt{1 - x^2} = \frac{-ik't}{\sqrt{1 - k'^2t^2}},$$

where the sign of $\sqrt{1 - x^2}$ is fixed as in § 137, since x varies from 1 to $\frac{1}{k}$ as t varies from 0 to 1. We have, then,
$$\frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}} = i \frac{dt}{\sqrt{(1 - t^2)(1 - k'^2t^2)}};$$

whence
$$K' = -i \int_1^{\frac{1}{k}} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}. \qquad (7)$$

From (1) and (7),
$$K + iK' = \int_0^{\frac{1}{k}} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}; \qquad (8)$$

whence
$$\begin{aligned} \text{sn}\,(K + iK') &= \frac{1}{k}, \\ \text{cn}\,(K + iK') &= \frac{-ik'}{k}, \\ \text{dn}\,(K + iK') &= 0, \end{aligned} \qquad (9)$$

where the sign of cn $(K + iK')$ is fixed as in § 137.

The use of the results (9) in the addition formulas gives
$$\begin{aligned} \text{sn}\,(u + K + iK') &= \frac{\text{dn}\,u}{k\,\text{cn}\,u}, \\ \text{cn}\,(u + K + iK') &= \frac{-ik'}{k\,\text{cn}\,u}, \\ \text{dn}\,(u + K + iK') &= \frac{ik'\,\text{sn}\,u}{\text{cn}\,u}; \end{aligned} \qquad (10)$$

whence we get
$$\begin{aligned} \text{sn}\,(u + 2K + 2iK') &= -\text{sn}\,u, \\ \text{cn}\,(u + 2K + 2iK') &= \text{cn}\,u, \\ \text{dn}\,(u + 2K + 2iK') &= -\text{dn}\,u. \end{aligned} \qquad (11)$$

We may also place
$$u + iK' = (u + K + iK') - K$$

and, applying the addition formula and making use of (2) and (10), together with (7), (8), (9), § 154, we get

$$\begin{aligned}\operatorname{sn}(u+iK') &= \frac{1}{k \operatorname{sn} u},\\ \operatorname{cn}(u+iK') &= -\frac{i \operatorname{dn} u}{k \operatorname{sn} u},\\ \operatorname{dn}(u+iK') &= \frac{-i \operatorname{cn} u}{\operatorname{sn} u};\end{aligned} \tag{12}$$

whence

$$\begin{aligned}\operatorname{sn}(u+2iK') &= \operatorname{sn} u,\\ \operatorname{cn}(u+2iK') &= -\operatorname{cn} u,\\ \operatorname{dn}(u+2iK') &= -\operatorname{dn} u,\end{aligned} \tag{13}$$

and

$$\begin{aligned}\operatorname{sn}(u+4iK') &= \operatorname{sn} u,\\ \operatorname{cn}(u+4iK') &= \operatorname{cn} u,\\ \operatorname{dn}(u+4iK') &= \operatorname{dn} u.\end{aligned} \tag{14}$$

Some of the results obtained may be summed up in the following theorem:

The elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, *are doubly periodic functions: the function* $\operatorname{sn} u$ *has the periods* $4K$ *and* $2iK'$; *the function* $\operatorname{cn} u$ *has the periods* $4K$ *and* $2K+2iK'$; *the function* $\operatorname{dn} u$ *has the periods* $2K$ *and* $4iK'$.

157. Limiting cases. CASE I. If we place $k = 0$, we have

$$u = \int_0^x \frac{dx}{\sqrt{1-x^2}};$$

whence $\operatorname{sn} u = \sin u, \quad \operatorname{cn} u = \cos u, \quad \operatorname{dn} u = 1.$

The quantity K becomes $\frac{\pi}{2}$, and the period $4K$ is 2π. The quantity K' becomes infinite and ceases to have a significance as a period.

CASE II. If we place $k = 1$, we have

$$u = \int_0^x \frac{dx}{1-x^2};$$

whence

$$\begin{aligned}\operatorname{sn} u &= \tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}},\\ \operatorname{cn} u &= \frac{2}{e^u + e^{-u}},\\ \operatorname{dn} u &= \frac{2}{e^u + 2e^{-u}}.\end{aligned}$$

C

The quantity $K' = \frac{\pi}{2}$, and the period $4\,iK'$ becomes $2\,\pi i$. The quantity K is infinite and ceases to be of importance.

158. Elliptic integrals in the complex plane. The periods of the elliptic functions may also be obtained by considering the various values acquired by the elliptic integral, now written as

$$w = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \tag{1}$$

by various paths in the plane of the complex variable $z = x + iy$.

The singular points of the function

$$f(z) = \frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}} \tag{2}$$

are $\pm 1, \pm \frac{1}{k}$. At all other points $f(z)$ is regular.

By § 145, any two paths of integration which do not include between them one or more of the singular points will give the same value of the integral. We may therefore examine the difference in the value of the integral for two paths which do surround one or more singular points, and since a passage around two singular points does not change the sign of $f(z)$ we shall take paths surrounding two singular points.

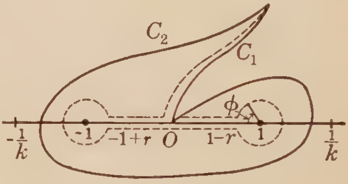

FIG. 124

Let w_0 be the value of w obtained by integration along any given path C_1 (Fig. 124); that is, let

$$w_0 = \int_{\substack{0\\(C_1)}}^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \tag{3}$$

and let C_2 be a path which, together with C_1, incloses the two singular points 1 and -1. The path C_2 may be deformed without changing the value of the integral into a portion of the axis of reals from 0 to $1-r$, a circle with radius r and center at $z = 1$, a portion of the axis of reals from $1-r$ to $-1+r$, a circle of radius r and center at $z = -1$, a portion of the axis of reals from $-1+r$ to 0, and the curve C_1. This is shown in the dotted line of the figure. The value of the integral is then equal to the sum of the following seven integrals, taken along these paths:

$$
\begin{aligned}
&\int_0^{1-r} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_{\circ} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \\
&\quad + \int_{1-r}^{0} \frac{dx}{-\sqrt{(1-x^2)(1-k^2x^2)}} + \int_0^{-1+r} \frac{dx}{-\sqrt{(1-x^2)(1-k^2x^2)}} \\
&\quad + \int_{\circ} \frac{dz}{-\sqrt{(1-z^2)(1-k^2z^2)}} + \int_{-1+r}^{0} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \\
&\quad + \int_{\substack{0 \\ (C_1)}}^{z} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},
\end{aligned}
\tag{4}
$$

where the second and fifth integrals are taken around the respective circles, and the changes of sign of the radical are due to passage around a singular point. Except for the integrals around the two small circles the sum of the integrals in (4) is clearly equal to

$$4\int_0^{1-r} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + w_0. \tag{5}$$

To evaluate the integral around the circle with center at $z=1$, take ϕ as in Fig. 124.

Then $$1 - z = 1 - x - yi = r(\cos\phi - i\sin\phi),$$
$$dz = r(\sin\phi + i\cos\phi)d\phi,$$
$$\int \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = r^{\frac{1}{2}} \int_0^{2\pi} F(\phi)d\phi,$$

where $F(\phi)$ does not contain r as a factor. A similar expression is obtained for the integral around the circle with center at $z = -1$.

Now let $r \to 0$. The value of the sum (4) does not depend upon r. We may therefore take the limit and have

$$\int_{\substack{0 \\ (C_2)}}^{z} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = 4K + w_0. \tag{6}$$

But the value of z in (3) and (6) is the same. Hence

$$\operatorname{sn}(4K + w_0) = \operatorname{sn} w_0. \tag{7}$$

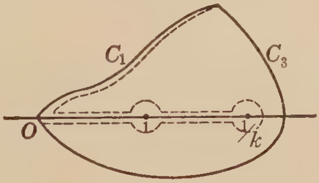

FIG. 125

Consider now a path C_3 (Fig. 125) which, together with C_1, incloses the two singular points 1 and $\frac{1}{k}\cdot$ It may be deformed into the path shown by the dotted line, and by the methods just used we have

$$\underset{(C_3)}{\int_0^z} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = 2\,iK' + w_0. \tag{8}$$

Hence

$$\operatorname{sn}(2\,iK' + w_0) = \operatorname{sn} w_0. \tag{9}$$

Similar discussions may be applied to other paths to obtain the formulas of § 156 and other like formulas.

Consider the path of integration (Fig. 126) consisting of the axis of imaginaries from O to iR, a semicircle of radius R, and the axis of imaginaries from $-iR$ to O. If R is taken greater than $\frac{1}{k}$, the path may be deformed into one along the axis of reals from 1 to $\frac{1}{k}$ and back, and therefore the value of the integral along this path is $2\,iK'$. Along the semicircle place

$$z = R(\cos\theta + i\sin\theta).$$

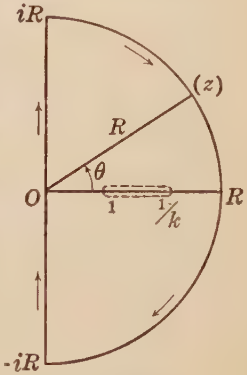

FIG. 126

The integral around the semicircle is then of the form

$$\frac{1}{R}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} F(\theta)d\theta,$$

where $F(\theta)$ remains finite as $R \rightarrow \infty$. Hence, by the limit process already employed, we have

$$2\int_0^\infty \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = 2\,iK',$$

or

$$\int_0^\infty \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = iK'. \tag{10}$$

In this demonstration the axis of imaginaries may be replaced by any curve running to infinity and symmetric about O without essential change, so that the path of integration in (10) need not be specified.

Equation (10) gives the result

$$\operatorname{sn}(iK') = \infty,$$

from which

$$\operatorname{cn}(iK') = \infty,$$

$$\operatorname{dn}(iK') = \infty.$$

From (12), § 156, and (4), § 154,

$$\begin{aligned}\operatorname{sn}(w+iK') &= \frac{1}{k \operatorname{sn} w}\\ &= \frac{1}{kw}\left[1-\frac{1+k^2}{6}w^2+\cdots\right]^{-1}\\ &= \frac{1}{kw}+\frac{1+k^2}{6\,k}w+\cdots,\end{aligned}$$

or, replacing $w+iK'$ by w,

$$\operatorname{sn} w = \frac{1}{k(w-iK')}+\frac{1+k^2}{6\,k}(w-iK')+\cdots,$$

which shows that at $w=iK'$ the function sn w has a simple pole with residue $\frac{1}{k}$.

Similarly, the function cn u has at $w=iK'$ a simple pole with residue $-\frac{i}{k}$, and the function dn w has a simple pole with residue $-i$.

159. Elliptic integrals of the second kind and of the third kind. We have defined

$$\int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx \tag{1}$$

as an elliptic integral of the second kind. If we place

$$x = \sin\phi,$$

the integral (1) becomes

$$E(k,\ \phi) = \int_0^\phi \sqrt{1-k^2\sin^2\phi}\,d\phi. \tag{2}$$

When $\phi=\frac{\pi}{2}$ in (2), the integral is denoted by E; thus,

$$E = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\phi}\,d\phi. \tag{3}$$

The values both of $E(k,\ \phi)$ and of E for various values of ϕ and k may be computed by expansions into power series, or such values may be found in tables.

From (2) we have

$$\begin{aligned}E(k,\ \phi+\pi) &= \int_0^{\phi+\pi} \sqrt{1-k^2\sin^2\phi}\,d\phi\\ &= \int_0^{\pi} \sqrt{1-k^2\sin^2\phi}\,d\phi + \int_\pi^{\phi+\pi} \sqrt{1-k^2\sin^2\phi}\,d\phi. \end{aligned}\tag{4}$$

The first integral is evidently $2E$, and if in the last integral we place

$$\phi = \psi + \pi,$$

it becomes

$$\int_0^\phi \sqrt{1 - k^2 \sin^2 \psi}\, d\psi = E(k, \phi).$$

Hence

$$E(k, \phi + \pi) = 2E + E(k, \phi). \tag{5}$$

This integral occurs in the problem of finding the length of an arc of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

for which we readily compute

$$s = \int_0^x \frac{\sqrt{a^2 - e^2x^2}}{\sqrt{a^2 - x^2}}\, dx, \tag{6}$$

where $e = \dfrac{\sqrt{a^2 - b^2}}{a}$ is the eccentricity of the ellipse. If in (6) we place

$$x = a \sin \phi,$$

it becomes

$$s = a \int_0^\phi \sqrt{1 - e^2 \sin^2 \phi}\, d\phi = aE(e, \phi).$$

Then aE is the length of a quarter-arc of the ellipse.

The integral (1) may be made to depend upon the elliptic function $\operatorname{sn} u$. For if we substitute

$$x = \operatorname{sn} u$$

and denote the result by $E(u)$, we have

$$E(u) = \int_0^u \operatorname{dn}^2 u\, du = u - k^2 \int_0^u \operatorname{sn}^2 u\, du. \tag{7}$$

This result may be expressed as a power series in u by the aid of § 154.

The elliptic integral of the third type has been written

$$\int_0^x \frac{dx}{(x^2 - a)\sqrt{(1 - x^2)(1 - k^2x^2)}}, \tag{8}$$

or

$$\int_0^\phi \frac{d\phi}{(\sin^2 \phi - a)\sqrt{1 - k^2 \sin^2 \phi}}.$$

If we place

$$x = \operatorname{sn} u, \qquad a = \operatorname{sn}^2 \alpha,$$

this becomes

$$\int_0^u \frac{du}{\operatorname{sn}^2 u - \operatorname{sn}^2 \alpha}. \tag{9}$$

A further study of the two integrals (1) and (2) would involve properties of doubly periodic functions, which lie outside the scope of this book.

160. The function $p(u)$. Just as the elliptic integral of the type (1), § 152, defines the function sn u, so the elliptic integral

$$u=\int_x^\infty \frac{dx}{\sqrt{4x^3-g_2x-g_3}}=\int_x^\infty \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}} \tag{1}$$

defines x as the function $x=p(u)$, (2)

which is the Weierstrass elliptic function. We note first that

$$\frac{dx}{du}=p'(u)=\sqrt{4p^3(u)-g_2p(u)-g_3},$$

so that $p(u)$ is a solution of the differential equation

$$\left(\frac{d\phi}{dx}\right)^2=4\phi^3-g_2\phi-g_3. \tag{3}$$

The integral (1) may be reduced to a Legendrian integral of the first kind. Let us place

$$x=e_3+\frac{g^2}{t^2}, \tag{4}$$

where g is to be determined later. Then, using the second form of the integral (1), we have

$$u=\frac{1}{g}\int_0^t \frac{dt}{\sqrt{\left(1-\frac{e_1-e_3}{g^2}t^2\right)\left(1-\frac{e_2-e_3}{g^2}t^2\right)}}; \tag{5}$$

and if we take $g^2=e_1-e_3$ and $k^2=\frac{e_2-e_3}{e_1-e_3}$, (6)

(5) becomes $u=\frac{1}{g}\int_0^t \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$. (7)

If e_1, e_2, e_3 are real and $e_1>e_2>e_3$, k in (6) is positive and less than unity.

We have, from (7), $t=\text{sn}\,(gu)$, (8)

and from (2) and (4) we have a relation existing between the function sn u and the function $p(u)$,

$$p(u)=e_3+\frac{g^2}{\text{sn}^2(gu)}. \tag{9}$$

Let us place $\omega=\frac{K}{g}$, $\omega'=\frac{K'}{g}$, where K and K' are derived from the integral in (7). Then, from (9) and the formulas of § 156, we have

$$\begin{aligned} p(u+2\omega)&=p(u),\\ p(u+2i\omega')&=p(u). \end{aligned} \tag{10}$$

Hence *the function* $p(u)$ *is a doubly periodic function with the two periods* $2\,\omega$ *and* $2\,i\omega'$.

From (9) we obtain readily

$$\begin{aligned} p(\omega) &= e_3 + g^2 = e_1, \\ p(\omega+\omega') &= e_3 + k^2g^2 = e_2, \\ p(\omega') &= e_3. \end{aligned} \tag{11}$$

Consequently, from (1),

$$\begin{aligned} \omega &= \int_{e_1}^{\infty} \frac{dx}{\sqrt{4\,x^3 - g_2x - g_3}}, \\ \omega' &= \int_{e_3}^{\infty} \frac{dx}{\sqrt{4\,x^3 - g_2x - g_3}}, \\ \omega+\omega' &= \int_{e_2}^{\infty} \frac{dx}{\sqrt{4\,x^3 - g_2x - g_3}}, \end{aligned} \tag{12}$$

and, by combining the last two,

$$\omega = \int_{e_2}^{e_3} \frac{dx}{\sqrt{4\,x^3 - g_2x - g_3}}. \tag{13}$$

161. Applications. 1. Consider the problem of finding the length of the arc of a lemniscate

$$r^2 = 2\,a^2 \cos 2\,\theta. \tag{1}$$

We have $$s = \int_0^r \frac{2\,a^2\,dr}{\sqrt{4\,a^4 - r^4}}. \tag{2}$$

Place $$r = \frac{a}{z^{\frac{1}{2}}}.$$

Then $$\frac{s}{a} = \int_z^{\infty} \frac{dz}{\sqrt{4\,z^3 - z}}; \tag{3}$$

whence $$\frac{a^2}{r^2} = z = p\left(\frac{s}{a}\right). \tag{4}$$

In the elliptic integral (3), $g_2 = 1$, $g_3 = 0$, $e_1 = \frac{1}{2}$, $e_2 = 0$, $e_3 = -\frac{1}{2}$.

$$\omega = \int_{\frac{1}{2}}^{\infty} \frac{dz}{\sqrt{4\,z^3 - z}} = \int_0^{a\sqrt{2}} \frac{2\,a\,dr}{\sqrt{4\,a^4 - r^4}}, \tag{5}$$

which is the length of a quarter of the lemniscate, and

$$\omega' = \int_{-\frac{1}{2}}^{\infty} \frac{dz}{\sqrt{4\,z^3 - z}} = \int_0^{a\sqrt{-2}} \frac{2\,a\,dr}{\sqrt{4\,a^4 - r^4}}, \tag{6}$$

which is obviously imaginary.

2. Consider the motion of a spherical pendulum, defined as a particle of mass m constrained to move on the surface of a sphere under the influence of gravity.

Using cylinder coördinates we take the equation of the sphere as

$$r^2 + z^2 = a^2. \tag{7}$$

Then $$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2 = \frac{a^2}{a^2 - z^2}\, dz^2 + (a^2 - z^2) d\theta^2. \tag{8}$$

We shall use Hamilton's principle and the Lagrangian equations with $q_1 = z$, $q_2 = \theta$. Then

$$T = \frac{m}{2} v^2 = \frac{m}{2}\left[\frac{a^2}{a^2 - z^2}\, \dot{z}^2 + (a^2 - z^2)\dot{\theta}^2\right], \tag{9}$$

$$V = mgz. \tag{10}$$

The Lagrangian equations are then

$$\frac{a^2 z \dot{z}^2}{(a^2 - z^2)^2} - z\dot{\theta}^2 - g - \frac{d}{dt}\left(\frac{a^2 \dot{z}}{a^2 - z^2}\right) = 0, \tag{11}$$

$$-\frac{d}{dt}[(a^2 - z^2)\dot{\theta}] = 0. \tag{12}$$

Equation (12) gives $$\dot{\theta} = \frac{C_1}{a^2 - z^2}. \tag{13}$$

Using this in (11) and carrying out the indicated differentiation, we have

$$\frac{a^2 \ddot{z}}{a^2 - z^2} + \frac{a^2 z \dot{z}^2}{(a^2 - z^2)^2} + \frac{C_1{}^2 z}{(a^2 - z^2)^2} + g = 0, \tag{14}$$

which may be written as

$$\frac{d}{dt}\left(\frac{a^2 \dot{z}^2}{a^2 - z^2}\right) + \frac{d}{dt}\left(\frac{C_1{}^2}{a^2 - z^2}\right) = -2\, g\dot{z}, \tag{15}$$

and integrating with respect to t, we have

$$\frac{a^2 \dot{z}^2}{a^2 - z^2} + \frac{C_1{}^2}{a^2 - z^2} = -2\, gz + C_2, \tag{16}$$

or $$a^2\left(\frac{dz}{dt}\right)^2 = (C_2 - 2\, gz)(a^2 - z^2) - C_1{}^2. \tag{17}$$

This has a resemblance to the differential equation (3), § 160, satisfied by the function $p(t)$, since the polynomial on the right is cubic. To reduce to the exact form of that equation we substitute

$$z = As + B$$

and determine A and B so that equation (17) becomes of the form

$$\left(\frac{ds}{dt}\right)^2 = 4\,s^3 - g_1 s - g_2. \tag{18}$$

This gives, by considering the coefficients of s^3 and s^2,

$$A = \frac{2\,a^2}{g}, \qquad B = \frac{C_2}{6\,g}; \tag{19}$$

whence g_1 and g_2 may be determined.

The cubic polynomial in (17) has three real roots. For if $z = z_0$, the initial position of the body, the cubic is $+$, since the velocity is real, and $z_0 < a$, since the particle is on the sphere. When $z = \infty$, the cubic is $+$; when $z = -a$, the cubic is $-$; when $z = z_0$, the cubic is $+$; when $z = a$, the cubic is $-$.

Hence the cubic has a root $z = z_1$ between $+\infty$ and a, a root $z = z_2$ between a and z_0, and a root $z = z_3$ between z_0 and $-a$. Moreover, the cubic is positive between z_2 and z_3 and hence these are the extreme heights of the particle.

Correspondingly, we may write (18) as

$$\left(\frac{ds}{dt}\right)^2 = 4(s - e_1)(s - e_2)(s - e_3), \tag{20}$$

where $\quad e_1 = \dfrac{z_1 - B}{A}, \qquad e_2 = \dfrac{z_2 - B}{A}, \qquad e_3 = \dfrac{z_3 - B}{A},$

and $\dfrac{ds}{dt}$ is real when s is between e_2 and e_3.

From (20) we have

$$t = \int_s^\infty \frac{ds}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}} + C. \tag{21}$$

To determine the constant C let us measure t from the time when $s = e_3$. Then

$$C = -\int_{e_3}^\infty \frac{ds}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}}; \tag{22}$$

and if we take $\quad u = \displaystyle\int_s^\infty \frac{ds}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}}$ (23)

we have, from (12), § 160,

$$C = -\,\omega'. \tag{24}$$

Then
$$u = t + \omega'; \tag{25}$$
whence
$$s = p(u) = p(t + \omega'),$$
and
$$z = Ap(t + \omega') + B. \tag{26}$$

When $z = z_3$, then $s = e_3$, and (21) with (24) gives
$$t = (\omega + \omega') - \omega' = \omega, \tag{27}$$
so that the half-period ω is the time that it takes the particle to go between the extreme positions z_2 and z_3.

EXERCISES

1. Show that $\int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$, where $k > 1$, can be reduced to a similar integral with $k < 1$.

2. Solve the pendulum problem when the bob goes completely around the circle, taking the velocity at the bottom of the path as v_0.

3. Solve the pendulum problem when the bob just reaches the top of the swing.

4. A skipping-rope revolves so that each element of the string has constant angular velocity about an axis. Assuming that on each element there act centrifugal force and the tension of the rope (neglecting gravity), find the equation of the curve in which the rope swings.

5. Show that
$$K = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \cdots\right]$$
with $k < 1$.

6. Show that
$$E = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \frac{1}{3}\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 - \frac{1}{5}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 - \cdots\right]$$
with $k < 1$.

7. Show that
$$\int_x^1 \frac{dx}{\sqrt{(1 - x^2)(k'^2 + k^2x^2)}} = \text{sn}^{-1}(\sqrt{1 - x^2}, k).$$

8. Show that
$$\int_0^x \frac{dx}{\sqrt{(1 + x^2)(1 + k'^2x^2)}} = \text{sn}^{-1}\left(\frac{x}{\sqrt{1 + x^2}}, k\right).$$

9. Show that
$$\int_0^x \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a}\,\text{sn}^{-1}\left(\frac{x}{b}, \frac{b}{a}\right).$$

10. Show that

$$\int_x^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a}\,\text{sn}^{-1}\left(\sqrt{\frac{a^2-x^2}{a^2-b^2}},\ \frac{\sqrt{a^2-b^2}}{a}\right).$$

11. If s is the length of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and e its eccentricity, show that

$$s = \frac{b^2}{ae}F(\phi, k) - aeE(\phi, k) + ae\tan\phi\sqrt{1-k^2\sin^2\phi},$$

where $k=\frac{1}{e}$.

12. Show that $$p(-u)=p(u).$$

13. Show that

$$\text{sn}\,(z+y)\,\text{sn}\,(z-y) = \frac{\text{sn}^2 z - \text{sn}^2 y}{1-k^2\,\text{sn}^2 z\,\text{sn}^2 y}.$$

14. Show that $$\text{sn}^2 x = \frac{1-\text{cn}\,2x}{1+\text{dn}\,2x}.$$

15. Verify the series expansion of § 154.

16. Find the formulas for sn $(u-v)$, cn $(u-v)$, and dn $(u-v)$.

17. Find the values of sn $(2K-u)$, cn $(2K-u)$, and dn $(2K-u)$.

18. Discuss the difference in the values of $\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$ corresponding to two paths which together inclose one of the points 1 or $\frac{1}{k}$.

19. Discuss the difference in the values of $\int_z^\infty \frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}$ corresponding to two paths which together inclose two of the points e_1, e_2, e_3.

20. Discuss the difference in the values of $\int_z^\infty \frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}$ corresponding to two paths which together inclose one of the points e_1, e_2, or e_3.

ANSWERS

CHAPTER I

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13. $x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots.$

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14. $x + \frac{2x^3}{3} + \frac{8x^5}{15} + \frac{16x^7}{35} + \cdots.$

15. $x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots.$

16. $x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \cdots.$

17. $-\sin a.$
18. $\log \frac{a}{b}.$
19. $-2.$
20. 2.
21. $\frac{1}{2}.$
22. $-\frac{5}{3}.$
23. $\infty.$
24. 0.
25. 0.
26. 0.
27. $0;\ \frac{a_0}{b_0};\ \infty.$
31. $\infty.$
32. 1.
33. 1.
34. $e^{ab}.$
35. 1.
36. Second.

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37. Fourth. 38. Second; third. 41. Second. 42. Second. 49. $\frac{d^2z}{dx^2} + 1 = 0.$

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50. $x\frac{d^2z}{dx^2} + (2n+1)\frac{dz}{dx} + xz = 0.$

51. $t\frac{d^2z}{dt^2} + (n+1)\frac{dz}{dt} + z = 0.$

52. $\frac{d^2y}{d\theta^2} + \cot\theta\frac{dy}{d\theta} - n(n+1)\theta = 0.$

CHAPTER II

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21. $(-1, 1).$
22. $(-\infty, \infty).$
23. $\left(-\frac{a}{b}, \frac{a}{b}\right).$
24. $(-2, 2).$
25. $(-\infty, \infty).$
26. $(-1, 1).$
27. $(-2, 2).$
28. $(-3, 3).$
29. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$

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37. $(-\infty, \infty).$ 38. $(-\infty, \infty).$ 39. $(-1, 1).$ 40. $(-1, 1).$ 41. $(-1, 1).$

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42. $x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots;\ (-1, 1).$

43. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots;\ (-1, 1).$

44. $x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots;\ (-1, 1).$

45. $x - \frac{x^3}{3} + \frac{1}{5}\frac{x^5}{2!} - \frac{1}{7}\frac{x^7}{3!} + \cdots;\ (-\infty, \infty)$.

46. $x - \frac{1}{5}\frac{x^5}{2!} + \frac{1}{9}\frac{x^9}{4!} - \frac{1}{13}\frac{x^{13}}{6!} + \cdots;\ (-\infty, \infty)$.

47. $\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \cdots;\ (-1, +1)$.

48. $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$.

49. $1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots$.

50. $\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots$.

51. $1 + x + x^2 + \frac{2x^3}{3} + \cdots$.

52. $x + \frac{2x^3}{3} + \frac{8x^5}{15} + \frac{16x^7}{35} + \cdots$.

53. $1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \cdots$.

54. $e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right)$.

55. $1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \cdots$.

56. $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$.

57. $x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$.

CHAPTER III

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20. $\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y}\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2 y}{dx^2}$.

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25. $x^2\frac{\partial^2 V}{\partial x^2} + 2xy\frac{\partial^2 V}{\partial x \partial y} + y^2\frac{\partial^2 V}{\partial y^2} + x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}$.

30. Circles; toward O; $\frac{2}{x^2 + y^2}$.

31. Away from O; $\frac{1}{\sqrt{x^2 + y^2}}$.

32. Parallel to OX; $\frac{1}{a}$.

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33. 60° with OX; 1.

39. $x^2 + y^2 - xy + x - y$.

40. $\frac{x^2y^2 - y^2 - 1}{2x^2}$.

41. $\sqrt{x^2 + y^2} - y$.

42. $\log(x^2 + y^2) - \tan^{-1}\frac{x}{y}$.

43. $\frac{1}{2}(x^2y^2z^2 - x^2z^2 + z^2y^2 - z^2)$.

44. $xy + yz + zx - (b + c)x - (c + a)y - (a + b)z$.

45. $(x + y)(y + z)(z + x)$.

46. $\frac{x}{y} + \frac{y}{z} + \frac{z}{x}$.

CHAPTER IV

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1. $-\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}};\ \frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}$.

2. $-\frac{x^{n-1}}{y^{n-1}};\ -\frac{(n-1)a^n x^{n-2}}{y^{2n-1}}$.

3. $-e^{-(x+y)};\ e^{2y-x}$.

4. $\frac{2x + y}{x - 2y};\ \frac{10(x^2 + y^2)}{(x - 2y)^3}$.

5. $-\frac{\tan x}{\tan y};\ -\frac{\sec^2 x \tan^2 y + \tan^2 x \sec^2 y}{\tan^3 y}$.

6. $\frac{y^2}{x(y - x)};\ \frac{y^2(y - 2x)}{x(y - x)^3}$.

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8. $-\dfrac{c^2x}{a^2z}$. 9. $\dfrac{kyz-1}{1-kxy}$. 10. $-\dfrac{x^2}{z^2}$. 11. $\dfrac{-2\,xz\sqrt{z^2-y^2}}{2\,z^3\sqrt{z^2-y^2}-y(x^2+y^2+z^2)}$.

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24. $\dfrac{v-x^2}{u^2-uv}$. 25. $\dfrac{y-u}{u-v}$. 28. $\left(\dfrac{\partial f}{\partial r}\right)^2-\dfrac{1}{r^2}\left(\dfrac{\partial f}{\partial \theta}\right)^2$.

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31. $\dfrac{u-v}{v-x}$. 32. $-\dfrac{ux+vy}{u^2+v^2}$. 33. $\dfrac{\dfrac{\partial f}{\partial u}+2\,x\dfrac{\partial f}{\partial v}}{\dfrac{\partial f}{\partial u}-2\,z\dfrac{\partial f}{\partial v}}$, $u=x+y-z$, $v=x^2-y^2+z^2$.

CHAPTER V

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2. $\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{\sqrt{2}}$. 5. $z+z_1=2\,ax_1x+2\,by_1y$. 6. $\dfrac{x_1x}{a^2}+\dfrac{y_1y}{b^2}=1$.

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11. $\cos^{-1}\dfrac{a^2-b^2+c^2}{2\,ac}$.

13. $\left(-\dfrac{ad}{a^2+b^2+c^2}, -\dfrac{bd}{a^2+b^2+c^2}, -\dfrac{cd}{a^2+b^2+c^2}\right)$.

14. (a, a, a), $(-a, -a, a)$, $(-a, a, -a)$, $(a, -a, -a)$.

15. Intersection of medians.

16. $V=\dfrac{8\,abc}{3\sqrt{3}}$.

17. $x=\dfrac{2\,aK}{a^2+b^2+c^2}$, $y=\dfrac{2\,bK}{a^2+b^2+c^2}$, $z=\dfrac{2\,cK}{a^2+b^2+c^2}$.

19. Length $=\sqrt{u}$, where $(ab-h^2)u^2-(a+b)cu+c^2=0$; $\theta=\dfrac{1}{2}\tan^{-1}\dfrac{2\,h}{a-b}$, where θ is angle made with OX.

20. $x=\dfrac{pN}{p+q+r}$, $y=\dfrac{qN}{p+q+r}$, $z=\dfrac{rN}{p+q+r}$.

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24. $\tan^{-1}\dfrac{a}{k}$. 26. $\tan^{-1}\dfrac{t}{\sqrt{1+k^2}}$. 28. $\dfrac{a^2+k^2}{a}$; $\dfrac{a^2+k^2}{k}$. 29. $\dfrac{1}{2\sqrt{1+k^2}}$.

CHAPTER VI

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8. $27.96\cdots$. 9. $.1325\cdots$. 10. $-.0950\cdots$.

17. $\pi\log\dfrac{1+\sqrt{1-\alpha^2}}{2}$. $(-1<\alpha<1)$ 18. $\log(\alpha+1)$. $(\alpha>-1)$

CHAPTER VII

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15. $\dfrac{\pi\sqrt{2}}{16}$. 18. $\dfrac{\pi a^3}{70}$; $\dfrac{21\,a}{128}$; $\dfrac{9\,a^3}{715}$.

17. $\dfrac{a^3}{90}$; $\dfrac{3\,a}{28}$; $\dfrac{a^5}{1890}$. 19. $\dfrac{\pi}{5}$; $\dfrac{5\,\Gamma(\frac{1}{4})}{21\sqrt{\pi}\,\Gamma(\frac{3}{4})}$; $\dfrac{2\,\pi}{15}$.

CHAPTER VIII

Page 201

1. $6\frac{1}{3}$; $6\frac{3}{20}$; $5\frac{5}{8}$.
2. 5; $6\frac{7}{20}$; 9.
3. -2; $-\frac{8}{3}-\pi$.
4. $\frac{3}{8}\pi a^2$.
5. $\frac{\pi a^2}{b}(3b-2a)$.
6. $\frac{\pi a^2}{b}(3b+2a)$.
7. $\left(2-\frac{\pi}{2}\right)a^2$.

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12. $8a^2$.
13. $2a^2(\pi-2)$.
14. $2\sqrt{2}\,\pi a^2$.
15. $\frac{a^2}{18}(20-3\pi)$.
16. $\frac{28\pi a^2}{9}$.
17. $4\pi\sqrt{5}$.
22. 0; 1.
23. 1; 1.

CHAPTER X

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1. $\sec x\tan y=c$.
2. $2\log xy+x^2-y^2=c$.
3. $x^2(1-y^2)=c(1+x^2)$.
4. $x^2+2xy=c$.
5. $x^2=c^2-2cy$.
6. $x=c\sin\frac{y}{x}$.
7. $x^2+4xy-y^2-6x-2y=c$.
8. $(x+y)^2+2y=c$.

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9. $y=\frac{1}{2}(\sin x+\cos x)+ce^{-x}$.
10. $xy=(x-1)e^x+c$.
11. $y=1+x^2+c\sqrt{1+x^2}$.
12. $4y^2+2x^2+2x+3+ce^{2x}$.
13. $(e^{\tan^{-1}x}+c)y=e^{\tan^{-1}x}$.
14. $(1+cx)y=x(1+x^2)$.
15. $1+y^2-x^2y^2=cx^2$.
16. $(x-y)^3-2y^3=c$.
17. $\log(1+x^2)+2xy=c$.
18. $e^{-\frac{x}{y}}-y=c$.
19. $xy+\log\frac{x}{y}=c$.
20. $3x^3\log x-y^3=cx^3$.
21. $y=a_0\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)$.
22. $y=a_0+\frac{k}{2}x^2$.
23. $y=a_0\left(1+\frac{x^2}{2}+\frac{x^4}{2^2\cdot 2!}+\frac{x^6}{2^3\cdot 3!}+\frac{x^8}{2^4\cdot 4!}+\cdots\right)$.
24. $y=a_0+a_0^2x+\frac{2a_0^3+1}{2}x^2+\frac{3a_0^4+a_0}{3}x^3+\frac{12a_0^5+5a_0^2}{12}x^4+\cdots$.
25. $y=a_0+a_0^2x+a_0^3x^2+a_0^4x^3+\frac{4a_0^5+1}{4}x^4+\cdots$.
26. $y=a_0+a_0^2x+\frac{2a_0^3-1}{2}x^2+\frac{3a_0^4-a_0}{3}x^3+\frac{12a_0^5-5a_0^2}{12}x^4+\cdots$.
27. $(x^2+y^2-c)(y-cx)=0$.
28. $(x^2y-c)(y-cx)=0$.
29. $y^2=4c(x+c)$.
30. $y^2=2cx-c^2$.
31. $y=c(x+c)^2$.
32. $x^2=\sin^2(y-c)$.
33. $cy^2-2c^2xy+c^3x^2=1$.
34. $c^2(1+y^2)-2c^3xy+c^4x^2=1$.
35. $(x-y)^2-2k(x+y)+k^2=0$.
36. $x^2+4ky-4k^2=0$.
37. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=k^{\frac{2}{3}}$.
38. $4xy=k^2$.
39. $16y^3+27x^4=0$.
40. $x^2-4y^2=0$.
41. $2xy=\pm k^2$.

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42. $4(x - 2\,a)^3 = 27\,ay^2$.
43. $3\,x^2 - 4\,xy = 0$.
44. $x^3 + (x + 2\,a)y^2 = 0$.
46. $27\,ay^2 = 4(x - 2\,a)^3$.
49. $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.
50. $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2\,a$.
51. $2\,x^2 + y^2 = c^2$.
52. $x^2 + y^2 - 2\,a^2 \log x = c$.
53. Self-orthogonal.
54. $y = cx^4$.
58. $r^2 = c \sin 2\,\theta$.
59. $r = c(1 - \cos \theta)$.

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60. $r = e^{\sqrt{c^2 - \theta^2}}$.
61. $x^2 + y^2 = a^2$.
62. $4\,x^2y + 1 = 0$.
63. $y^2 = a^2$.
64. $xy^2 - 4\,a^3 = 0$.
65. $27\,y = 4\,x^3$.
66. $xy + yz + zx - (b + c)x - (c + a)y - (a + b)z = k$.
67. $x^2z + y^2x + z^2y = cxyz$.
68. $(z + x)y + c(y + z) = 0$.
69. $xz + y^2 - yz \log cz = 0$.
70. $\log z = \tan^{-1} \frac{y}{x} + c$.
71. $x^2 - y^2 = c_1$, $y^2 - z^2 = c_2$.
72. $x = z + c_1$, $y^2 = 2\,zx + c_2$.
73. $x^2 + z^2 = c_1{}^2$, $y + c_2 = \frac{c_1}{2} \log \frac{z - c_1}{z + c_1}$.
74. $x + y = c_1e^z$, $2\,y = z^2 + c_2$.
75. $x^2 - y^2 = c_1y$, $x - y = c_2z$.
76. $x + y + z = c_1$, $x^2 + y^2 + z^2 = c_2$.
77. $x + y + 2\,z = c_1$, $x - y = c_2z^2$.
78. $y = c_1z$, $x^2 + y^2 + z^2 = c_2z$.

CHAPTER XI

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1. $y = c_1 \sin x + [c_2 - \log (\sec x + \tan x)] \cos x$.
2. $y = (x + c_1) \sin x + (c_2 + \log \cos x) \cos x$.
3. $y = [c_1 + \log(1 - x)]e^x + [c_2 - \log(1 - x)]xe^x$.
4. $y = c_1x + c_2e^x + x^2 + 1$.
5. $y = c_1e^{2x} + c_2e^{-4x} + \frac{1}{10} \sin 2\,x - \frac{3}{10} \cos 2\,x$.
6. $y = c_1e^{3x} + c_2e^{-5x} - \frac{1}{2}$.
7. $y = c_1 + c_2e^x + \frac{1}{2}\,x^3 - \frac{3}{4}\,x^2 + \frac{5}{4}\,x$.
8. $y = (c_1 + c_2x + \frac{1}{2}\,x^2)e^x + \frac{5}{9}\,e^{2x}$.
9. $y = e^x(c_1 \cos x\sqrt{2} + c_2 \sin x\sqrt{2}) + \frac{1}{3}\,x^2 + \frac{4}{9}\,x + \frac{2}{27}$.
10. $y = (c_1 + c_2x)e^{-3x} + \frac{e^{3x}}{200}(4 \sin 2\,x - 3 \cos 2\,x)$.

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11. $y = c_1 \cos 3\,x + c_2 \sin 3\,x + \frac{x}{37}(6 \sin 3\,x + \cos 3\,x)$.
12. $y = e^{-\frac{x}{2}}\left(c_1 \cos \frac{x\sqrt{3}}{2} + c_2 \sin \frac{x\sqrt{3}}{2}\right) + 2 + \frac{1}{13}\,e^{3x}$.
13. $y = c_1 + c_2x + c_3 \cos x + c_4 \sin x + \frac{3\,x^2}{2}$.
14. $y = c_1e^{-x} + e^{-\frac{x}{2}}\left(c_2 \cos \frac{x\sqrt{3}}{2} + c_2 \sin \frac{x\sqrt{3}}{2}\right) + \frac{1}{2}(\cos x - \sin x)$.
15. $y = c_1 + \left(c_2 + c_3x + \frac{x^2}{4}\right)e^{2x} + \frac{1}{4}\,x$.
16. $y = c_1x + c_2x^6 - \frac{1}{4}\,x^2$.
17. $y = c_1x^2 + \frac{c_2}{x^3} - \frac{1}{4}\,x \log x - \frac{3}{16}\,x$.
18. $y = c_1 + c_2 \log x + c_3(\log x)^2 + \frac{1}{8}\,x^2$.
19. $x = y = ce^{-\frac{t}{2}}$.

20. $x = c_1e^{-3t} + c_2e^{2t} - \frac{1}{6}e^{-t} - \frac{5}{26}\sin 2t - \frac{1}{26}\cos 2t,$
$y = -3c_1e^{-3t} - \frac{c_2}{2}e^{2t} - \frac{1}{6}e^{-t} + \frac{17}{52}\sin 2t + \frac{19}{52}\cos 2t.$

21. $x = c_1 + c_2e^{t} + c_3e^{-2t} + \frac{1}{4}t^2 - \frac{3}{4}t,$
$y = 2c_1 + c_2e^{t} - 2c_3e^{-2t} - \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{2}.$

22. $y = \frac{x\sqrt{2}}{2}\sqrt{c_1{}^2 - x^2} + \frac{\sqrt{2}\,c_1{}^2}{2}\sin^{-1}\frac{x}{c_1} + c_2.$

23. $y = c_1x^3 + c_2.$

24. $y = \frac{c_1}{2}\log(x - a) - \frac{(x-a)^2}{4c_1} + c_2.$

25. $(y + a)^2 = c_1x + c_2.$

26. $y = c_1 \tanh \frac{x + 2y + c_2}{2c_1}.$

27. $y = c_1 \tanh c_1(x + c_2).$

28. $y = c_1 \cosh \frac{x + c_2}{c_1}.$

29. $y = c_1x\sin 2x + c_2x\cos 2x + \frac{x}{8}e^{2x}.$

30. $y = \frac{1}{x}\left(c_1\cos x + c_2\sin x - \frac{1}{3}\sin 2x\right).$

31. $y = \left(c_1x^3 + \frac{c_2}{x^3}\right)e^{x}.$

32. $y = (c_1e^{x} + c_2e^{-x})\cos x - \frac{1}{2}\sin 2x.$

33. $y = c_1e^{-e^{x}} + c_2e^{-2e^{x}} + 1.$

34. $y = (c_1 + c_2\cos x)e^{\cos x} - 5 - 4\cos x - \cos^2 x.$

35. $y = a_0\sum_{k=0}^{k=\infty}\frac{(-1)^kx^{2k}}{(2k)!} + a_1\sum_{k=0}^{k=\infty}\frac{(-1)^kx^{2k+1}}{(2k+1)!}.$

36. $$y = a_0\left[1 + \sum_{k=1}^{k=\infty}\frac{n(n-4)(n-16)\cdots[n-(2k-2)^2]}{(2k)!}x^{2k}\right]$$
$$+ a_1\left[x + \sum_{k=1}^{k=\infty}\frac{(n-1)(n-9)(x-25)\cdots[n-(2k-1)^2]}{(2k+1)!}x^{2k+1}\right].$$

37. $$y = a_0\left[1 + \sum_{k=1}^{k=\infty}(-1)^k\frac{1\cdot 4\cdot 7\cdots(3k-2)}{(3k)!}n^kx^{3k}\right]$$
$$+ a_1\left[x + \sum_{k=1}^{k=\infty}(-1)^k\frac{2\cdot 5\cdot 8\cdots(3k-1)}{(3k+1)!}n^kx^{3k+1}\right].$$

38. $y = \frac{a_0}{x} + a_1\sum_{k=0}^{k=\infty}(-1)^k\frac{3}{(k+3)k!}x^k.$

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39. $y = a_0(6 - 4x + x^2) + \frac{a_1}{x^2}(1 - 4x + 6x^2 - 4x^3 + x^4).$

42. $\frac{a^2}{\sqrt{k}}.$ 43. $\frac{2a^{\frac{4}{3}}}{\sqrt{3k}}.$ 45. $\frac{a}{2}\sqrt{3k}.$ 46. $\frac{1}{6}(4 - \sqrt{2})\sqrt{a^3}.$

CHAPTER XIII

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1. $z = \phi_1(y)e^{ax} + \phi_2(y)e^{-ax}$.
2. $z = \dfrac{x^{a+1}}{a+1}\phi_1(y) + \phi_2(y)$.
3. $z = \phi_1(y) + \phi_2(y)e^{ax}$.
4. $z = \phi_1(y)e^{x} + \phi_2(y)e^{2x}$.
5. $z = [\phi_1 y) + x\phi_2(y)]e^{x}$.
6. $z = \phi_1(y)e^{x} + \phi_2(y)e^{-x} - y$.
7. $z = \dfrac{x^4y^4}{12} + x\phi_1(y) + \phi_2(y)$.
8. $\phi(x - y, y - z) = 0$.
9. $\phi(x^2 + y^2, z) = 0$.
10. $\phi\left(\dfrac{y}{x}, \dfrac{z^2}{x}\right) = 0$.
11. $\phi(x + y + z, x^2 + y^2 - z^2) = 0$.
12. $\phi\left(x^2 + y^2, z + \tan^{-1}\dfrac{y}{x}\right) = 0$.
13. $\phi[(2x - y)e^{-3z}, (4x + y)e^{-9z}] = 0$.
14. $\phi\left(\dfrac{y}{x}, \dfrac{z}{x}\right) = 0$.

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21. $2\left(\dfrac{\sin x}{1} - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \dfrac{\sin 4x}{4} + \cdots\right)$.

22. $\dfrac{\pi^2}{3} - 4\left(\dfrac{\cos x}{1^2} - \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} - \cdots\right)$.

23. $\dfrac{2}{\pi}\left[\left(\dfrac{\pi^3}{1} - \dfrac{6\pi}{1^3}\right)\sin x - \left(\dfrac{\pi^3}{2} - \dfrac{6\pi}{2^3}\right)\sin 2x + \left(\dfrac{\pi^3}{3} - \dfrac{6\pi}{3^3}\right)\sin 3x - \cdots\right]$.

24. $\dfrac{\pi}{2} + 2\left(\dfrac{\sin x}{1} + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \cdots\right)$.

25. $\dfrac{\pi}{4} - \dfrac{2}{\pi}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^3} + \cdots\right) - \left(\dfrac{\sin x}{1} - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \cdots\right)$.

26. $$\frac{3\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + \left(\frac{3\sin x}{1} - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right).$$

27. $$\frac{\pi^2}{6} - 2\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right) + \frac{1}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x - \frac{\pi^2}{4}\sin 4x + \cdots\right].$$

28. $\pi - 2\left(\dfrac{\sin x}{1} + \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} + \dfrac{\sin 4x}{4} + \cdots\right)$.

29. $$\frac{4\pi^2}{3} + 4\pi\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right) - 4\pi\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right).$$

30. $$2\pi^3 + 12\pi\left(\frac{\cos x}{1} + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots\right) + 4\left[\left(\frac{3}{1^3} - \frac{2\pi^2}{1}\right)\sin x + \left(\frac{3}{2^3} - \frac{2\pi^2}{2}\right)\sin 2x + \left(\frac{3}{3^3} - \frac{2\pi^2}{3}\right)\sin 3x + \cdots\right].$$

31. $\dfrac{1}{2} + \dfrac{2}{\pi}\left(\dfrac{\sin x}{1} + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \cdots\right)$.

32. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \cdots\right)$.

33. $\dfrac{4}{\pi}\left(\dfrac{\sin x}{1} + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \cdots\right)$.

34. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \cdots\right)$.

35. $\frac{2}{\pi}\left[\left(\frac{\pi^2}{1}-\frac{4}{1^3}\right)\sin x-\frac{\pi^2}{2}\sin 2\,x+\left(\frac{\pi^2}{3}-\frac{4}{3^3}\right)\sin 3\,x-\frac{\pi^2}{4}\sin 4\,x+\cdots\right].$

36. $\frac{4}{\pi}\left(\frac{1}{2}-\frac{\cos 2\,x}{1\cdot 3}-\frac{\cos 4\,x}{3\cdot 5}-\frac{\cos 6\,x}{5\cdot 7}-\cdots\right).$

37. $\frac{4}{\pi}\left(\frac{2\sin 2\,x}{2^2-1}+\frac{4\sin 4\,x}{4^2-1}+\frac{6\sin 6\,x}{6^2-1}+\cdots\right).$

38. $\frac{c_1}{r}+c_2.$

39. $c_1\log r+c_2.$

40. $\sum e^{-k^2t}\left(a_k\cos\frac{kx}{a}+b_k\sin\frac{k}{a}\right).$

41. $\sum(A_k\cos kx+B_k\sin kx)(C_k\cos ky+D_k\sin ky)e^{-\frac{k^2}{h^2}t}.$

42. $\frac{4}{\pi}\left(r\sin\theta+\frac{r^3}{3}\sin 3\,\theta+\frac{r^5}{5}\sin 5\,\theta+\cdots\right).$

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43. $\frac{4}{\pi}\left(e^{-\frac{a^2\pi^2t}{c^2}}\sin\frac{\pi x}{c}+\frac{1}{3}e^{-\frac{9\,a^2\pi^2t}{c^2}}\sin\frac{3\,\pi x}{c}+\frac{1}{5}e^{-\frac{25\,a^2\pi^2t}{c^2}}\sin\frac{5\,\pi x}{c}+\cdots\right).$

44. $\frac{1}{2}-\frac{2}{\pi}\left(r\sin\theta+\frac{r^3}{3}\sin 3\,\theta+\frac{r^5}{5}\sin 5\,\theta+\cdots\right).$

45. $\sum e^{-(b^2+h^2k^2)t}(A_k\cos kx+B_k\sin kx).$

46. $\sum A_k\sin\frac{k\pi x}{l}\cos\frac{k\pi t}{l}.$

47. $z=AJ_0[2\sqrt{k(l-x)}].$

48. $y=AJ_0\left(\frac{nx}{\sqrt{gh}}\right)\cos nt.$

49. $y=AJ_0\left(2\sqrt{\frac{nx}{\sqrt{gk}}}\right)\cos nt.$

50. $$\frac{2\,M}{a^2}\left[\frac{1}{2}\cdot\frac{a}{r}-\frac{1}{2}\cdot\frac{1}{4}\frac{a^3}{r^3}P_2(\cos\theta)\right.$$
$$\left.+\frac{1}{2}\cdot\frac{1\cdot 3}{4\cdot 6}\frac{a^5}{r^5}P_4(\cos\theta)-\frac{1}{2}\cdot\frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\frac{a^7}{r^7}P_6(\cos\theta)+\cdots\right],\ (r>a)$$
$$\frac{2\,M}{a}\left[1\mp\frac{r}{a}P_1(\cos\theta)+\frac{1}{2}\frac{r^2}{a^2}P_2\,(\cos\theta)\right.$$
$$\left.-\frac{1}{2}\cdot\frac{1}{4}\frac{r^4}{a^4}P_4(\cos\theta)+\frac{1}{2}\cdot\frac{1\cdot 3}{4\cdot 6}\frac{r^6}{a^6}P_6(\cos\theta)+\cdots\right],\ (r<a)$$
where the second term is minus when $\theta<\frac{\pi}{2}$ and positive when $\theta>\frac{\pi}{2}$.

CHAPTER XIV

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1. $y=c_1x+c_2.$
2. $r\cos\,(\theta-c_2)=c_1.$
3. $\sin\,(\theta-c_2)=c_1\cot\phi.$
6. $\theta-c_2=\int\frac{c_1dr}{\sqrt{(r^2+k^2)(r^2+k^2-c_1)}}.$
7. $r\sqrt{r^2-c_1{}^2}\,d\theta=c_1{}^2\sqrt{1+f'^2(r)}\,dr.$
8. Cycloid.
9. $g(x-c_2)=\sqrt{2\,g(y+a)-c_1{}^2}.$
10. Catenary.

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11. Circle.
12. Circle.
13. Catenary.

CHAPTER XV

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36. $\frac{\pi}{a^4}\left(1-\frac{ma+2}{2}e^{-ma}\right).$

37. $\frac{\pi}{\sqrt{2}}e^{-\frac{m}{\sqrt{2}}}\left(\cos\frac{m}{\sqrt{2}}+\sin\frac{m}{\sqrt{2}}\right).$

38. $\frac{\pi}{\sin a\pi}.$

39. $\pi(\cot ax-\cot bx).$

40. $\frac{\pi}{2\,a}e^{-a}.$

41. $\frac{1}{2}\sqrt{\frac{\pi}{2}}.$

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$e^{i\phi} = \text{Cos}\,\theta + i \sin \theta$

m_1 = imaginary = $\alpha + i\beta$ = root of quadrati equation

If $\alpha + i\beta$ is one root $\alpha - i\beta$ is the Conjugate.